Final Project (T I T)

Due date: Feb. 12, 2010

Consider the boundary-value problem

$$(*)\quad \begin{cases} d\Delta u + u(m(x) - u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplace operator, $m(x)$ is a nonconstant smooth function, and Ω is a bounded smooth domain in $\mathbb{R}^n$ with ν as its unit outer normal. Assume that $\int_\Omega m(x)\,dx \geq 0$. Then, prove the following statements:

(1) For every $d > 0$, $(*)$ has a unique positive solution u_d which is globally ~~asymptot~~ asymptotically stable. (Observe that we do not need to assume that "$m(x)$ changes sign in Ω" for this fact to hold.)

(2) $u_d \to \overline{m} \equiv \frac{1}{|\Omega|}\int_\Omega m$ as $d \to \infty$.

(3) $u_d \to m_+$ (the positive part of m) as $d \to 0$.

(4) Setting $P(d) \equiv \int_\Omega u_d(x)\,dx$, we see that, in the case $m > 0$ in Ω, from (2) and (3) above,

$$P(d) \to \int_\Omega m \quad \text{as } d \to 0 \text{ or } \infty.$$

Prove that, for every $d > 0$, $P(d) > \int_\Omega m$.

(**) Can you locate the maximum point of $P(d)$ in terms of m? (Open)