# Circle Packing and Teichmüller Space

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## Contents



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# 1 Introduction

A circle on the complex plane $\mathbb{C}$ is defined as either an usual circle or a straight line in complex analysis. This definition is very understandable if we look at a circle on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Riemann sphere can be identified with the complex projective line, and every projective transformation on $\widehat{\mathbb{C}}$ through the identification sends a circle to a circle. Conversely, a transformation of $\widehat{\mathbb{C}}$ which sends a circle to a circle turns out to be projective. Thus the 1-dimensional complex projective geometry fits well with the concept of a circle.

To globalize the notion of a circle, consider a surface $S$, a real 2-dimensional manifold, locally modeled on $\widehat{\mathbb{C}}$ such that any coordinate change is a restriction of a projective transformation. Such a geometric structure is referred as a projective structure, and we call a surface with a projective structure simply a projective Riemann surface. A 1-dimensional subset in $S$ is said to be a circle if its developed image on $\widehat{\mathbb{C}}$ is a circle.

Now, we are interested in a circle packing on a projective Riemann surface, which will be a particular configuration of circles such that all complementary regions are curvilinear triangles. It enjoys both rigid and flexible properties in connection with Teichmüller spaces. The main purpose of this chapter is to discuss such interesting properties from geometric viewpoints.

Recall that there are three typical Riemannian geometries in dimension 2, that is, the spherical, euclidean and hyperbolic geometries. They are geometries with constant curvature 1, 0 and $-1$ respectively, and regarded as subgeometry of the 1-deimensional complex projective geometry. In particular, a constant curvature surface is a projective Riemann surface.

The rigidity we discuss here is by Koebe [8], Anderee [1] and Thurston [17] for realization of a circle packing on a constant curvature surface. More specifically, we describe in rather uniform way how a combinatorial adjacency data of circles determines uniquely a constant curvature surface which supports a geometric packing with prescribed data.

This rigidity motivated Brooks [3, 4] to analyze flexible nature when we allow to have quadrilateral complementary regions. He succeeded to parameterize the deformation in terms of continued fractional type numerical invariants, and deduced the density of packable constant curvature surfaces in the Teichmüller space. We discuss Brooks' idea briefly, and see how his parameter works through quasi-conformal deformation theory.

On the other hand, extending the problem Koebe-Andreev-Thurston settled on constant curvature surfaces, one may ask what the set of projective Riemann surfaces supporting a circle packing with a common combinatorial data looks like. It leads us to analyze flexibility of the object in question. According to [9, 10, 11], we present here a construction of the moduli space of pairs of such projective Riemann surfaces with circle packings, and see its basic properties. In particular, we discuss our belief that the moduli space provides a sort of uniformization in terms of circle packing. We state it as a conjecture in more explicit form, and report some progress towards it.

The organization of this chapter is as follows. Reviewing the basics of the subject in the next section, we discuss rigidity results together with density on constant curvature surfaces due to Koebe-Andreev, Thurston and Brooks. We then discuss flexibility by constructing moduli spaces, and formulate a conjecture along with some supporting evidence.

## 2 Circle Packing

### 2.1 Circle on the Riemann Sphere

The complex projective line, which is the space of complex lines through the origin in $\mathbb{C}^2$, can be identified with the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by assigning slopes. Then, $\widehat{\mathbb{C}}$ can be identified by the stereographic projection with the unit sphere $\mathbb{S}^2$ in the 3-dimensional euclidean space $\mathbb{E}^3$. These are the main playgrounds on which the circle is placed. Note that $\widehat{\mathbb{C}}$ admits a natural orientation coming from its complex structure.

Given four complex numbers $a$, $b$, $c$, $d$ such that $ad - bc \neq 0$, we obtain a linear fractional transformation,

$$z \longmapsto \frac{az+b}{cz+d},$$

which acts on $\widehat{\mathbb{C}}$ as a projective transformation. Such projective transformations form a group isomorphic to $\mathrm{PGL}(2,\mathbb{C}) \cong \mathrm{PSL}(2,\mathbb{C})$ by identifying a transformation with a matrix consisting of these four numbers. The action of the projective linear group preserves the orientation.

Another important transformation on $\widehat{\mathbb{C}}$ is a complex conjugation,

$$z \longmapsto \bar{z},$$

which fixes $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and reverses the orientation. A complex conjugation and $\mathrm{PGL}(2,\mathbb{C})$ generate a group Möb of Möbius transformations which fits into a splitting short exact sequence,

$$1 \to \mathrm{PGL}(2,\mathbb{C}) \to \text{Möb} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

A circle is, by definition, the image of $\widehat{\mathbb{R}}$ by a projective transformation. Through the identification of $\widehat{\mathbb{C}}$ with $\mathbb{S}^2$, a circle on $\widehat{\mathbb{C}}$ projects to either a straight line or a circle in the usual sense on $\mathbb{C}$.

A circle on $\widehat{\mathbb{C}} = \mathbb{S}^2$ can be defined also as a metric circle with respect to the spherical metric. However, note that a projective transformation, which sends a circle to a circle, does not preserve the spherical metric in general.

## 2.2 Projective Geometry

The pair $(\mathrm{PGL}(2,\mathbb{C}), \widehat{\mathbb{C}})$ of the $2 \times 2$ projective linear group and the Riemann sphere is called the 1-dimensional complex projective geometry in the sprit of Felix Klein's Erlangen program. The projective geometry contains three typical 2-dimensional geometries as subgeometry.

The unitary group $\mathrm{U}(2)$ in $\mathrm{GL}(2,\mathbb{C})$ becomes $\mathrm{PU}(2) \subset \mathrm{PGL}(2,\mathbb{C})$ in the quotient. It is isomorphic to $\mathrm{SO}(3)$, and the pair $(\mathrm{PU}(2), \widehat{\mathbb{C}}) = (\mathrm{SO}(3), \mathbb{S}^2)$ is the spherical geometry.

The upper triangular group $\mathrm{UT}(2)$ in $\mathrm{GL}(2,\mathbb{C})$ yields $\mathrm{PUT}(2) \subset \mathrm{PGL}(2,\mathbb{C})$ whose action on $\widehat{\mathbb{C}}$ fixes $\{\infty\}$. It is isomorphic to the 1-dimensional complex affine transformation group $\mathrm{A}(1)$, and the pair $(\mathrm{PUT}(2), \mathbb{C}) = (\mathrm{A}(1), \mathbb{C})$ is the complex affine geometry. Under the canonical identification of $\mathbb{C}$ with the euclidean plane $\mathbb{E}^2$, the orientation preserving euclidean isometry group $\mathrm{Isom}_+ \mathbb{E}^2$ can be embedded in $\mathrm{A}(1)$ and the pair $(\mathrm{Isom}_+ \mathbb{E}^2, \mathbb{E}^2) \subset (\mathrm{A}(1), \mathbb{C})$ is the euclidean geomerty.

The action of the subgroup $\mathrm{U}(1,1)$ of $\mathrm{GL}(2,\mathbb{C})$, which preserves Hermitian form of signature $(1,1)$, leaves the unit disk $\mathbf{D}$ of $\mathbb{C}$ invariant. The projectivisation defines the hyperbolic geometry $(\mathrm{PU}(1,1), \mathbf{D})$, where the action of $\mathrm{PU}(1,1)$ preserves the orientation.

The spherical, euclidean and hyperbolic geometries have compact stabilizers and each admits a riemaniann metric of constant curvature, say 1, 0, $-1$ respectively, invariant under the action of transformation groups. A circle on $\widehat{\mathbb{C}}$ contained in the domain of these three geometries will be a metric circle in their own metrics.

## 2.3 Projective Riemann Surface

Let $\Sigma_g$ be a compact oriented surface of genus $g \geq 0$. A complex 1-dimensional projective structure, or simply a projective structure, on $\Sigma_g$ is a system of local coordinates compatible with the orientation modeled on the Riemann sphere such that on any two overlapping coordinate patches, the change of coordinates is a restriction of a projective transformation. In modern language, it is a geometric structure modeled on $(\mathrm{PGL}(2,\mathbb{C}), \widehat{\mathbb{C}})$.

Since projective transformations are holomorphic, every projective structure determines an underlying complex structure, and hence a surface with a projective structure can be regarded as a Riemann surface. For short, a surface with a projective structure will be called a *projective Riemann surface*.

Notice that projective structure is finer than complex structure, and different projective Riemann surfaces can share the same underlying complex structure.

Also, since the projective geometry contains the spherical, euclidean and hyperbolic geometries as subgeometry, any constant curvature surface is a projective Riemann surface and in particular has an underlying complex structure.

Let $S$ be a projective Riemann surface homeomorphic to $\Sigma_g$. We always attach to $S$ an orientation preserving homeomorphism,

$$h : \Sigma_g \longrightarrow S,$$

which we call a marking. Two projective Riemann surfaces, say $(S_1, h_1)$ and $(S_2, h_2)$, are considered to be marked projectively equivalent if there exists a projective isomorphism,

$$\varphi : S_1 \longrightarrow S_2,$$

such that $\varphi \circ h_1$ is homotopic to $h_2$. Since $\varphi \circ h_1$ is required to be homotopic to $h_2$, the marking determines the homotopy class of a projective isomorphism.

To each projective Riemann surface $S$, one assigns a developing map,

$$D : \widetilde{S} \longrightarrow \widehat{\mathbb{C}},$$

defined as an analytic continuation of a preferred local coordinate, where $\widetilde{S}$ is the universal cover of $S$. It is well defined up to composition with projective transformations. One also assigns to $S$ a holonomy representation,

$$\rho : \pi_1(S) \longrightarrow \mathrm{PGL}(2, \mathbb{C}),$$

defined so that the equivariance condition,

$$D(\gamma x) = \rho(\gamma) D(x),$$

holds for all $\gamma \in \pi_1(S)$ and $x \in \tilde{S}$, where $\pi_1(S)$ acts as deck transformations on $\tilde{S}$. It is well defined up to conjugation by projective transformations.

Let $\mathcal{T}_g$ be the Teichmüller space of $\Sigma_g$, namely the space of all complex structures on $\Sigma_g$ up to marked biholomorphic equivalence. $\mathcal{T}_g$ is homeomorphic to a real euclidean space of dimension 0, 2, $6g - 6$ according to whether $g =$ 0, 1 or otherwise. By the uniformization theorem, every Riemann surface is biholomorphic to a constant curvature surface, and hence every biholomorphic class is represented by a projective Riemann surface, but not uniquely. To see how many projective structures can share the same complex structure, we introduce the analytic viewpoint of projective structures below.

A holomorphic quadratic differential, $q = q(z)dz^2$, on a Riemann surface $R$ is an assignment of a holomorphic function $q(z)$ to each local coordinate $z$ such that if $z_1$ and $z_2$ are local coordinates with common domain, then

$$q_1(z_1) = q_2(z_2) \left( \frac{dz_2}{dz_1} \right)^2 .$$

In other words, it is a holomorphic section of the square of the holomorphic cotangent bundle (the canonical line bundle) of $R$. The set of all holomorphic quadratic differentials on $R$ becomes a complex vector space of complex dimension 0, 1, $3g-3$ according to whether $g = 0$, 1 or otherwise. The dimension count is deduced from the Riemann-Roch theorem.

Suppose we have a holomorphic quadratic differential $q$ on $R$. In a local coordinate $z$, the solutions of the Schwarzian differential equation,

$$2w''(z) + \frac{1}{2}q(z)w(z) = 0$$

forms a two-dimensional complex vector space. Then the ratio $\phi$ of two linearly independent solutions satisfies the identity,

$$\left(\frac{\phi''(z)}{\phi'(z)}\right)' - \frac{1}{2}\left(\frac{\phi''(z)}{\phi'(z)}\right)^2 = q(z),$$

where the left-hand side is called Schwarzian derivative of $\phi$. By the standard existence and uniqueness of solutions to systems of holomorphic differential equations, $\phi$ extends by analytic continuation to a holomorphic map,

$$D : \widetilde{R} \longrightarrow \widehat{\mathbb{C}},$$

of the universal cover of $R$ unique up to composition with a projective transformation. This map can be seen as a developing map of a projective structure on $R$ associated with $q$. Thus we have obtained a projective structure from a pair $(R, q)$ of a Riemann surface $R$ and a holomorphic quadratic differential $q$ on $R$. Conversely, if we are given a projective Riemann surface $S$, then the Schwarzian derivative of its developing map defines a holomorphic quadratic differential $q$ with respect to the underlying complex structure on $S$.

Hence the set of projective structures on $\Sigma_g$ corresponds bijectively to the set of all pairs $(R, q)$ where $R$ is a Riemann surface homeomorphic to $\Sigma_g$ and $q$ is a holomorphic quadratic differential on $R$.

Let $\mathcal{P}_g$ be the space of all projective structures on $\Sigma_g$ up to marked projective equivalence, in other words, the set of all projective Riemann surfaces homeomorphic to $\Sigma_g$ with marking. We have a natural projection,

$$\pi : \mathcal{P}_g \longrightarrow \mathcal{T}_g,$$

by assigning an underlying complex structure to each projective Riemann surface. This is a vector bundle of complex rank 0, $1, 3g-3$ according to whether $g = 0$, 1 or otherwise.

When $g = 0$, $\mathcal{P}_0$ and $\mathcal{T}_0$ both consist of a single point and the situation is quite simple. When $g \geq 1$, by the uniformization theorem, for each biholomorphic class of projective Riemann surfaces homeomorphic to $\Sigma_g$, there is a unique representative by either an euclidean torus or a hyperbolic surface according to whether $g = 1$ or $g \geq 2$. Hence, we obtain a natural section,

$$s : \mathcal{T}_g \longrightarrow \mathcal{P}_g,$$

to the projection $\pi : \mathcal{P}_g \to \mathcal{T}_g$ by assigning a corresponding constant curvature surface with marking.

In the case $g = 1$, there is a slight difference between complex affine structures and projective structures on $\Sigma_1$. Let $\mathcal{A}_1$ be the space of all complex affine structures on $\Sigma_1$. The image of a holonomy representation of a complex affine structure on $\Sigma_1$ which is not an euclidean structure is contained in the subgroup of A(1) whose action fixes $\{0, \infty\}$. This subgroup is invariant under an involutive conjugation induced by the transformation $z \mapsto 1/z$. The action defines a double cover,

$$\mathcal{A}_1 \longrightarrow \mathcal{P}_1,$$

branched along $s(\mathcal{T}_1)$. Hence the correspondence between complex affine structures and projective structures on the torus is generically two to one.

## 2.4 Circle Packing on Surface

A projective Riemann surface $S$ would be the most general underlying space when we discuss circle packings on a compact surface, since the group of projective transformations is the maximal group which sends a circle on the Riemann sphere to a circle.

**Definition 2.1.** A *circle* on a projective Riemann surface $S$ will be a homotopically trivial simple closed curve on $S$ whose developed image is a circle on $\widehat{\mathbb{C}}$.

A homotopically trivial simple closed curve on $\Sigma_g$ always bounds a disk. When a projective Riemann surface $S$ has genus $g \geq 1$, any circle on $S$ bounds a unique disk. However a circle on the Riemann sphere bounds disks in both sides. Hence in this particular case, we need to choose a bounding disk to each circle.

**Definition 2.2.** A *circle configuration* $C$ on a projective Riemann surface $S$ is a collection of circles such that there is an assignment of bounding disks to each member which are disjoint. Note that the assignment is unique if any when $C$ contains more then one members. We use the notation $C$ also for a subset of $S$.

Two circles on $\widehat{\mathbb{C}}$ bounding disjoint open disks either touch at a single point or coincide unless they are disjoint. However this is only for $\widehat{\mathbb{C}}$. For circle configurations on a projective Riemann surface of genus $g \geq 1$, two circles may touch at several points, and even a single circle may have self contacts.

**Definition 2.3.** To each circle configuration $C$ on a projective Riemann surface $S$, we assign a graph $\tau$ on $S$ and simltaneously on $\Sigma_g$ through a marking where vertices correspond to the circles of $C$ and two (or possibly one) vertices are joined by an edge for each point of tangency. We call $\tau$ a *nerve* of $C$.

A circle configuration $C$ determines an isotopy class of $\tau$ on $S$ and therefore on $\Sigma_g$ through a marking. We use only this topological property for $\tau$ and do not concern any geometric properties which $\tau$ may have.

Suppose we are given a circle configuration $C$ on some projective Riemann surface. If the complementary region of the union of bounding disks contains a non simply connected component, we can insert finitely many circles without changing the original configuration to make the complement simply connected. If a simply connected complementary region is bounded by a curvilinear polygon with more than four sides, then again we can insert finitely many circles to make the configuration have the property that the complementary regions consists of only curvilinear triangles and quadrilaterals. In this case, the nerve $\tau$ defines a cell decomposition of $\Sigma_g$ by triangles and quadrilaterals. This simplification of the shape of complementary regions is achieved elementarily. Thus we will always suppose that a circle configuration has this property from now on unless otherwise stated.

On the other hand, most of curvilinear quadrilateral regions cannot be filled by finitely many circles with only triangular complementary regions. In other words, every projective Riemann surface admits a circle configuration so that the complementary regions are only curvilinear triangles or quadrilaterals, but not every one admits a configuration with only triangular complementary regions. Thus, Brooks gave a special name for a circle configuration with this strong property in [4].

**Definition 2.4.** A circle configuration is said to be a *circle packing* if the complementary regions all are triangular, namely the nerve defines a cell decomposition of $\Sigma_g$ only by triangles.

The nerve here defines a triangulation of $\Sigma_g$ in the most general sense, namely some 1-simplex and hence 2-simplex may be immersed. Here are a few examples of circle packings illustrated in figures.

Figure 1 is the stereographic image of a circle packing on $\widehat{\mathbb{C}}$ whose nerve decomposes $\widehat{\mathbb{C}}$ as a tetrahedron. The circle packing $P$ on $\mathbb{E}^2$ pictured in Figure 2 is called a hexagonal packing. It can be seen also as a universal cover of some circle packing on the torus by taking a quotient of the group action generated by appropriate parallel translations preserving $P$. If we choose the maximal such group, we get the circle packing on the hexagonal torus by one circle with three self contact points.

A circle packing by one circle is realized also on a hyperbolic surface of genus $g \geq 2$. It has $3(2g-1)$ self contact points. Figure 3 represents a universal cover of such packing when $g = 2$. As we will discuss in §3, such circle packings admit a deformation. Figure 4 illustrates a small deformation of a circle packing in Figure 3 with the same combinatorics.

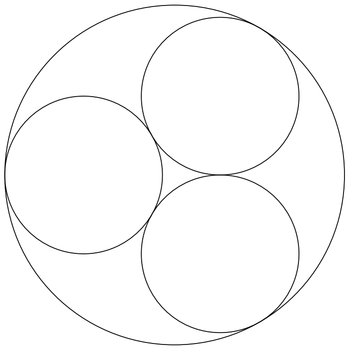

Figure 1.Tetrahedral packing

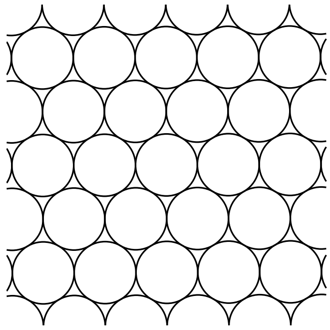

Figure 2.Hexagonal packing

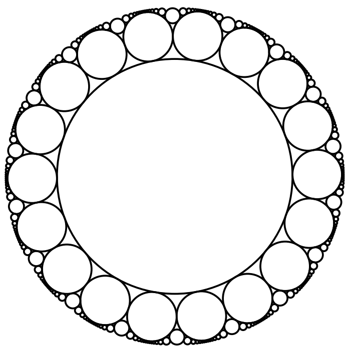

Figure 3.
Universal cover of a circle packing on a hyperbolic surface

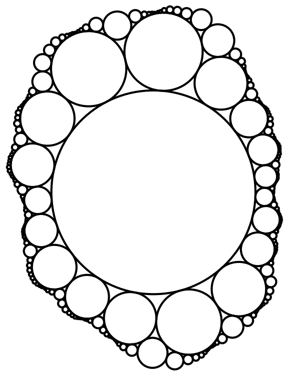

Figure 4.
Deformation of a circle packing in Figure 3

The graph which appears as a nerve of some circle configuration with polygonal complementary regions on a projective Riemann surface has the property that it defines a cell decomposition of the universal cover such that evely closed cell is embedded. For instance, if all cells are triangular, it defines a honest triangulation in the universal cover. We give a special name for such graphs.

**Definition 2.5.** A graph on $\Sigma_g$ is *simple* if it defines a cell decomposition in the universal cover such that each closed cell is embedded. In other words, if it defines a simple planer graph on the universal cover in the graph theoretic sense, namely no loops an no multiple edges.

A fundamental problem in the study of circle configurations on compact surfaces would be to understand the moduli space of the pairs $(S, C)$ of a projective Riemann surface $S$ and a circle configuration $C$ on $S$ under combinatorial control coming from the nerve. The pairs $(S, C)$ and $(S', C')$ will be equivalent if there is a projective isomorphism $\varphi : S \to S'$ compatible with marking such that $\varphi(C) = C'$.

**Problem 2.6.** Given a simple graph $\tau$ on $\Sigma_g$, find the moduli space of all pairs $(S, C)$ of a projective Riemann surface $S$ and a circle configuration $C$ on $S$ with a nerve isotopic to $\tau$ through a marking up to equivalence.

Our main concern will be when $\tau$ defines a cell decomposition by only triangles and quadrilaterals.

# 3 Rigidity

## 3.1 Rigidity Theorems

Consider a circle packing $P$ on the Riemann sphere $\widehat{\mathbb{C}}$ with a nerve $\tau$. Since any projective transformation $\varphi$ of $\widehat{\mathbb{C}}$ is isotopic to the identity and sends a circle to a circle, $P' = \varphi(P)$ defines also a circle packing on $\widehat{\mathbb{C}}$ with a nerve isotopic to $\tau$. Hence, $(\widehat{\mathbb{C}}, P)$ is equivalent to $(\widehat{\mathbb{C}}, P')$. Thus roughly speaking, the circle packing on $\widehat{\mathbb{C}}$ dominated by $\tau$ has a complex 3-dimensional freedom to move on $\widehat{\mathbb{C}}$. But this will be the only freedom. The projective rigidity of circle packings on the Riemann sphere was proved originally by Koebe and then rediscovered by Andreev.

**Theorem 3.1** (Koebe [8] , Andreev [1]). *Suppose a simple graph $\tau$ on $\Sigma_0$ defines a honest triangulation. Then there is a circle packing $P$ on the Riemann sphere $\widehat{\mathbb{C}}$, such that the nerve of $P$ is isotopic to $\tau$. Moreover, for any two such packings $P$ and $P'$, there is a projective transformation $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi(P) = P'$.*

Since every orientation preserving self homeomorphism of the Riemann sphere is isotopic to the identity, The map $\varphi$ in the theorem above may not be unique. In fact, every graph automorphism of $\tau$ which extends to an orientation preserving self homeomorphism of $\widehat{\mathbb{C}}$ can be realized by the restriction of a projective transformation.

When $g \geq 1$, the marking will be involved in the rigidity. Remember that there is a preferred homotopy class of a projective isomorphism between two surfaces with marking. Theorem 3.1 was generalized for higher genus surfaces by Thurston as marked projective rigidity within constant curvature structures. To see the difference of the results between the cases $g = 1$ and $g \geq 2$ carefully, we split the statement into two theorems.

**Theorem 3.2** (Thurston [17]). *Suppose a simple graph $\tau$ on $\Sigma_1$ defines a cell decomposition by triangles. Then there is an euclidean torus $S$ with marking and a circle packing $P$ on $S$, such that the nerve of $P$ is isotopic to $\tau$. Moreover, for any two such realizations $(S, P)$ and $(S', P')$, there is a projective isomorphism $\varphi : S \to S'$ compatible with marking such that $\varphi(P) = P'$.*

A projective isomorphism between euclidean tori with marking is either a contraction, an expansion or a parallel translation. Hence the realization of a marked euclidean structure on the torus here is unique up to scaling.

When we fix an euclidean structure, the parallel transformation, which is isotopic to the identity, moves a circle packing. Hence the circle packing on an euclidean torus controlled by $\tau$ has a complex 1-dimensional freedom to move.

Also, $\varphi$ in the theorem above may not be unique. In fact as in the spherical case, every graph automorphism of $\tau$ which extends to a self homeomorphism of $\Sigma_1$ isotopic to the identity can be realized by the restriction of a projective transformation.

When $g \geq 2$, we have

**Theorem 3.3** (Thurston [17]). *Suppose a simple graph $\tau$ on $\Sigma_g$ $(g \geq 2)$ defines a cell decomposition by triangles. Then there is a unique hyperbolic surface $S$ with marking and a unique circle packing $P$ on $S$, such that the nerve of $P$ is isotopic to $\tau$.*

Thus in this case, the combinatorial structure of $\tau$ completely determines the hyperbolic surface $S$ with marking and the location of a circle packing $P$ on $S$.

If we combine three theorems above, the rigidity will be established for the constant curvature surfaces up to marked projective equivalence. The unified statement is as follows.

**Theorem 3.4.** *Suppose a simple graph $\tau$ on $\Sigma_g$ $(g \geq 0)$ defines a cell decomposition by triangles. Then there is a constant curvature surface $S$ with marking and a circle packing $P$ on $S$, such that the nerve of $P$ is isotopic to $\tau$. Moreover, for any two such realizations $(S, P)$ and $(S', P')$, there is a projective isomorphism $\varphi : S \to S'$ compatible with marking such that $\varphi(P) = P'$.*

## 3.2 Unified Proof

The theorems in the previous subsection can be proved uniformly by an argument due to Thurston [17]. The hyperbolic case is its source.

To see this, let $\tau$ be a simple graph on $\Sigma_g\,(g \geq 2)$ which gives a cell decomposition by triangles, and $V_\tau$ the set of vertices. Let

$$r : V_\tau \longrightarrow \mathbb{R}_+$$

be any positive real valued function of the vertex set. It will be an assignment of radii to each vertex. Given $r \in \mathbb{R}^{V_\tau}$, and suppose three vertices $u, v, w \in V_\tau$ span a triangle, then $r(u)+r(v)$, $r(v)+r(w)$, $r(w)+r(u)$ satisfy the triangle inequality. Assigning a hyperbolic triangle with those side lengths to each triple $u, v, w$ which span a triangle on $\Sigma_g$, pasting these triangles along edges according to a cell decomposition defined by $\tau$, and we get a hyperbolic surface homeomorphic to $\Sigma_g$ with cone singularities at vertices. It admits a circle packing with centers at $V_\tau$ and radii $r$.

To each vertex, assign the curvature concentrated, and we obtain a curvature concentration map

$$\varrho_r : V_\tau \longrightarrow \mathbb{R},$$

where the value at $v \in V_\tau$ is equal to $2\pi -$ sum of angles meeting at $v$. The value at $v = 0$ if and only if $v$ is not singular. Thus if the trivial map denoted by 0 in $\mathbb{R}^{V_\tau}$, which has no curvature concentraton for any vertices, is uniquely attained by some radii assignment, then we are done.

Thurston regards the correspondence $r \mapsto \varrho_r$ as a map

$$\mu : \mathbb{R}_+^{V_\tau} \longrightarrow \mathbb{R}^{V_\tau},$$

and sets up the problem more globally. He then shows that $\mu$ is injective onto its image by comparison of the images of different $r$'s based on the Gauss-Bonnet formula. Moreover it follows that the image of $\mu$ contains 0 by looking at the asymptotic nature of $\mu$ together with an invariance of the domain. Thurston's analysis actually provides much more information about the map $\mu$, but in particular it established that $\mu^{-1}(0)$ gives the unique nonsingular hyperbolic surface with a circle packing whose nerve is isotopic to $\tau$.

When $g = 1$, an assignment of radii $r : V_\tau \to \mathbb{R}_+$, a curvature concentration $\varrho_r : V_\tau \to \mathbb{R}$ and the map $\mu : \mathbb{R}_+^{V_\tau} \to \mathbb{R}^{V_\tau}$ can be defined without any change. However, since homothetic radii give the same curvature concentration, $\mu$ will not be injective. Also the sum of curvature concentration at each vertex must be zero because of the Gauss-Bonnet formula, and thus $\mu$ can never be locally surjective.

To rule out such redundancy coming from expansion and contraction on the source and the target, we let

$$\Lambda_1 = \{r \in \mathbb{R}_+^{V_\tau} \mid \sum_{v \in V_\tau} r(v) = 1\}.$$

Then by a similar argument in the case $g \geq 2$, $\mu$ restricted to $\Lambda_1$ is shown to be injective onto its image in $\{\varrho \in \mathbb{R}^{V_\tau} \mid \sum_{v \in V_\tau} \varrho(v) = 0\}$. Moreover, the image contains 0.

Finally, the spherical case is reduced to the euclidean case. Choose three vertices $u, v, w \in V_\tau$ which span a triangle on $\widehat{\mathbb{C}}$, and locate $u, v, w$ so that they span an equilateral triangle on $\mathbb{C}$ with side length 2 and the other part of $\tau$ is contained in this triangle. Letting

$$\Lambda_0 = \{r \in \mathbb{R}_+^{V_\tau} \mid r(u) = r(v) = r(w) = 1\},$$

do the same construction of a singular euclidean surface for each $r \in \Lambda_0$ with a fixed triangle boundary spanned by $u, v$ and $w$. Let $V_0$ be the set of vertices other than $u, v, w$, namely $V_0 = V_\tau - \{u, v, w\}$. Then, the map

$$\mu : \Lambda_0 \longrightarrow \mathbb{R}^{V_0}$$

in this case becomes again injective and the image contains 0. The circle configuration on $\mathbb{C}$ corresponding to $\mu^{-1}(0)$ is pulled back to a circle packing on $\widehat{\mathbb{C}}$ with nerve $\tau$ by the stereographic projection.

**Remark 3.5.** The setup by Thurston above has led to a variational approach to find the solution $\mu^{-1}(0)$ with respect to the sup norm of $\varrho_r$ by Colin de Verdière in [5]. Bennett and Luo took another variational viewpoint in [2] based on combinatorial Ricci flow.

Since, for given $\tau$, the realization of the pair $(S, P)$ of a constant curvature surface $S$ and a circle packing $P$ on $S$ is unique up to marked projective equivalence, we give a special name to them.

**Definition 3.6.** We call the unique pair $(S, P)$ provided by the rigidity theorems in the previous subsection a *KAT solution*.

## 3.3 Density

One of conclusions of the rigidity results is that the number of constant curvature surfaces which admit a circle packing is at most countable, because the number of cell decompositions by triangles on $\Sigma_g$ is countable. In contrast with such a sparse situation, Brooks showed that such structures are dense in the Teichmüller space if the combinatorial control is ignored. We here briefly review his idea.

Brooks starts with a circle configuration on the Riemann sphere by four circles with two quadrilateral complementary regions. Let $Q$ be one quadrilateral complementary region normalized as located in a bounded part in Figure 5. The other quadrilateral region is unbounded in this normalization. Then one adds a unique circle which is tangent to either the top, left and bottom sides, or is tangent to the left, top and right sides. The dotted circle in Figure 5 is the one we add. Brooks called the former case as in Figure 5 a horizontal circle and the latter a vertical circle. Notice that adding this new circles cuts out two new triangles and one new quadrilateral, except in the rare case that this circle is tangent to all four sides.

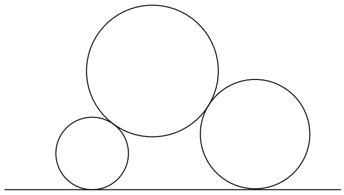

Figure 5. Normalized configuration by four circles

Now, iterate this process, each time adding a circle to the new quadrilateral created in the previous step. Denote by $n_1$ the number of horizontal circles obtained until one adds a vertical circle, $n_2$ the number of vertical circles then obtained until one adds a horizontal circles, and so on, and we consider the continued fraction expansion

$$c(Q) = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}$$

Note that the value $c(Q)$ depends on which circle we put on the horizontal line in the normalized picture. Hence, when we discuss about $c(Q)$, we remember the reference circle for $Q$.

The number $c(Q)$ is shown to vary continuously as the original four circles are varied in [3]. The rational number corresponds to a continued fraction which terminates in finitely many steps. Geometrically, this means that a circle configuration can be completed by a circle packing in $Q$ by inserting finitely many circles. Otherwise, the process of inserting circles in $Q$ never ends. The number $c(Q)$ is a primitive numerical projective invariant for a curvilinear quadrilateral on $\widehat{\mathbb{C}}$, which Brooks call a *continued fractional parameter*.

He globalizes the argument above to a circle configuration on the Riemann sphere $\widehat{\mathbb{C}}$ in [3]. Let $\tau$ be a simple graph on $\widehat{\mathbb{C}}$ which defines a cell decomposition only by triangles and quadrilaterals, and $Q_\tau$ the set of quadrilateral cells in the cell decomposition defined by $\tau$. Now, let $\mathcal{C}_\tau$ be the set of circle configurations on $\widehat{\mathbb{C}}$ with nerve isotopic to $\tau$ up to projective equivalence, endowed with a natural topology. The nerve $\tau$ is a combinatorial object as before and $\mathcal{C}_\tau$ is the moduli space. If there are no quadrilateral complementary regions in the circle configuration, $\mathcal{C}_\tau$ is just a point by Theorem 3.1. In general, we have

**Theorem 3.7** (Brooks [3]). *Suppose a simple graph $\tau$ on $\widehat{\mathbb{C}}$ defines a cell decomposition by triangles and quadrilateral, and let $Q_\tau$ be the set of quadrilateral cells. Then the map,*

$$c_\tau : \mathcal{C}_\tau \longrightarrow \mathbb{R}_+^{Q_\tau},$$

*assigning to each configuration $C \in \mathcal{C}_\tau$ continued fractional parameters of each member of $Q_\tau$, is a homeomorphism.*

This theorem can be understood in terms of languages of the quasi-conformal deformation theory. To see this, let us quickly review the theory. A finitely generated discrete subgroup $\Gamma$ of Möb is called a Kleinian group. The maximal region of $\widehat{\mathbb{C}}$ on which $\Gamma$ acts properly discontinuously is said to be a domain of discontinuity and denoted by $\Omega_\Gamma$. The quotient $\Omega_\Gamma/\Gamma$ is a Riemann surface of finite type by Ahlfors' finiteness theorem. A Kleinian group $\Gamma'$ is quasi-conformally equivalent to $\Gamma$ if there is a quasi-conformal map $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\Gamma' = \psi\Gamma\psi^{-1}$. Note here that $\Gamma'$ is assumed to be a Kleinian group and thus a discrete subgroup of Möb. A culminating result of the extensive quasi-conformal deformation theory developed by Ahlfors, Bers, Maskit, Marden and many others gives an explicit description of the deformation space as follows, see [14, 15] for the final form due to Sullivan.

**Theorem 3.8.** *The set $\mathcal{QC}(\Gamma)$ of quasi-conformal deformations of $\Gamma$ is homeomorphic to the Teichmüller space of the underlying topological surface of $\Omega_\Gamma/\Gamma$, in other words, the space of Riemann surfaces homeomorphic to $\Omega_\Gamma/\Gamma$ with marking up to marked biholomorphic equivalence.*

Let us come back to circle packing problem. A circle configuration $C$ was on $\widehat{\mathbb{C}}$ with only triangular and quadrilateral complementary regions. Let $\Gamma$ be a group generated by reflections about members in $C$. The group $\Gamma$ will be a Kleinian group and $\Omega_\Gamma/\Gamma$ in this case consists of finitely many triangles and quadrilaterals which can be identified with the complementary regions of $C$. The Teichmüller spaces of those regions are homeomorphic to a point and $\mathbb{R}$ respectively. Thus $\mathcal{QC}(\Gamma)$ in this case is homeomorphic to the euclidean space of dimension equal to the number of quadrilateral complementary regions.

Now, notice that there is a rigidity property of reflections in quasi-conformal deformations. If $\gamma \in \Gamma$ is a reflection, then the corresponding element $\gamma' \in \Gamma' = \psi\Gamma\psi^{-1}$ must be also a reflection, since a Möbius transformation which fixes a 1-dimensional set is a reflection.

Thus any quasi-conformal deformation $\Gamma' \in \mathcal{QC}(\Gamma)$ starting from a circle configuration $C$ defines a circle configuration $C'$ whose nerve is isotopic to $\tau$. Therefore, the theory provides a homeomorphic correspondence

$$\delta : \mathcal{QC}(\Gamma) \longrightarrow \mathcal{C}_\tau$$

and gives a new parameterizatioin of the Teichmüller space of quadrilateral regions, which is the underlying topological surface of $\Omega_\Gamma/\Gamma$, in terms of continued fractions.

**Corollary 3.9** (Brooks [3]). *The composition* $c_\tau \circ \delta : \mathcal{QC}(\Gamma) \to \mathbb{R}_+^{Q_\tau}$ *is a homeomorphism.*

This whole story can be extended to the study of circle packings on compact projective Riemann surfaces of genus $g \geq 2$. Let $S$ be a compact hyperbolic surface, $\Delta$ the image of a holonomy representation of $\pi_1(S)$ in $\mathrm{PGL}(2,\mathbb{C})$ and $C$ a circle configuration on $S$ with only triangular and quadrilateral complementary regions. Notice that $\Delta$ is a Fuchsian group. A circle configuration $C$ defines $\widetilde{C}$ on the universal cover $\widetilde{S} \subset \widehat{\mathbb{C}}$ with infinitely many members, but finitely many conjugacy classes with respect to the action of $\pi_1(S)$. Also, since $S$ is a hyperbolic surface, the boundary of $\widetilde{S}$ defines a circle $C_0 \subset \widehat{\mathbb{C}}$. Let $\Gamma$ be a group generated by $\Delta$ and reflections about members of $\widetilde{C}$ and $C_0$. Since the number of conjugacy classes of circles in $\widetilde{C}$ by the action of $\pi_1(S)$ is finite, $\Gamma$ is a finitely generated Kleinian group.

The deformation space $\mathcal{QC}(\Gamma)$ is homeomorphic to the Teichmüller space of the union of quadrilateral complementary regions of $C \subset S$ by quasi-conformal deformation theory, which is parameterized by continued fractional parameters by an equivariant version of Theorem 3.7.

Since the quasi-conformal deformation $\Gamma' \in \mathcal{QC}(\Gamma)$ contains a reflection about the boundary of the universal cover of a deformed surface, the result becomes again a hyperbolic surface. We can choose rational valued continued fractional parameter arbitrary close to the parameter of $C$. Then the quadrilateral regions in the deformed configuration $C'$ on a hyperbolic surface $S'$ becomes to be completed by a circle packing by inserting finitely many circles. Since a hyperbolic surface $S'$ could be chosen arbitrarily close to the original $S$, we get a twofold result by regarding $\mathcal{T}_g$ as the space of Riemann surfaces with marking up to biholomorphic equivalence, and also as the space of hyperbolic surfaces with marking up to isometry through the identification by the section $s : \mathcal{T}_g \to \mathcal{P}_g$.

**Theorem 3.10** (Brooks [4]). *The set of Riemann surfaces which admit a circle packing is dense in* $\mathcal{T}_g$. *Equivalently, the set of hyperbolic surfaces which admit a circle packing is dense in* $s(\mathcal{T}_g)$.

The argument so far depends on the well-developed theory of Kleinian groups, discrete subgroups of Möb. On the other hand, the image of the holonomy representation of a projective Riemann surface is not discrete in general, and some difficulty for analyzing density arises.

When $g = 1$, as a biproduct of the study of one circle packing on complex affine tori in [12], Mizushima proved the density of circle packing structures in $\mathcal{A}_1$. Since $\mathcal{A}_1$ doubly covers $\mathcal{P}_1$ branched along $s(\mathcal{T}_1)$, we have

**Theorem 3.11** (Mizushima [12]). *The set of projective Riemann tori which admit a circle packing is dense in* $\mathcal{P}_1$.

We may ask

**Question 3.12.** Is the set of projective Riemann surfaces which admit a circle packing dense in $\mathcal{P}_g$ for $g \geq 2$ ?

# 4 Flexibility

## 4.1 Constructing Moduli

Despite of the rigidity discussed in the previous section, circle packings on projective Riemann surfaces with combinatorial control by $\tau$ have a more flexible nature. It is expected to see the structure of a moduli space for the pairs $(S, P)$ of a projective Riemann surface $S$ and a circle packing $P$ on $S$ such that the nerve of $P$ is isotopic to $\tau$. To do this, a projective invariant of a circle packing on projective Riemann surfaces based on the cross ratio is introduced in [9]. In this section, we briefly review it based on the description in [11].

Suppose that $(S, P)$ is a pair of a projective Riemann surface $S$ and a circle packing $P$ on $S$. To each edge $e$ of the nerve $\tau$, we choose a lift $\widetilde{e}$ in $\widetilde{\tau}$ and associate a configuration of four circles on $\widehat{\mathbb{C}}$ in the developed image about $D(\widetilde{e})$, see Figure 6. Recall that the cross ratio of four distinct ordered points in $\widehat{\mathbb{C}}$ is given by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

It is the value of the image of $z_1$ under the projective transformation which takes $z_2, z_3$ and $z_4$ to $1, 0$ and $\infty$ respectively. The value assigned to the edge $e$ will be the imaginary part of the cross ratio of the four contact points $(p_{14}, p_{23}, p_{12}, p_{13})$ of the configuration chosen as in Figure 6 with orientation convention. The cross ratio of these four points is always purely imaginary with positive imaginary part.

Since the cross ratio is a projective invariant, the value does not depend on the choice of lift $\widetilde{e}$ and the developing map. Collecting the values for each edge, we obtain the map $\mathbf{x}$ of the edge set $E_\tau$ of $\tau$,

$$\mathbf{x} : E_\tau \longrightarrow \mathbb{R},$$

which is called a *cross ratio parameter*. The cross ratio of the edge $e$ determines the position of the circle $C_4$ in Figure 6 once the positions of $C_1, C_2$ and $C_3$ are fixed, and if the cross ratio of $e$ approaches $\infty$, then $C_4$ approaches $p_{13}$.

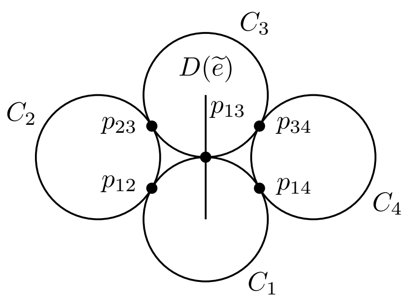


Figure 6. Four circle configuration

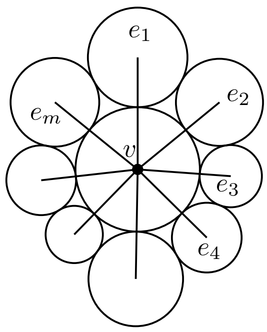


Figure 7. Surrounding circles

Obviously, not all real valued maps of $E_\tau$ can be a cross ratio parameter for some circle packing. To obtain necessary conditions, consider a normalized picture of a circle with its surrounding circles. The normalization we chose maps the central circle to the real line and one of the adjoining interstices to the standard interstice with vertices at $\infty$, 0 and $\sqrt{-1}$. This leads one to introduce an associated matrix $A \in \mathrm{SL}(2, \mathbb{R})$ to each edge $e \in E_\tau$. If the value of a cross ratio parameter at $e$ is $x$, $A$ is defined to be $\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

Then a simple computation shows that the associated matrix $A$ represents a transformation which sends the left triangular interstice of this configuration to the right triangular interstice.

Let $v$ be a vertex of $\tau$ with valence $m$. We read off the edges $e_1, \cdots, e_m$ incident to $v$ in a clockwise direction to obtain a sequence of assigned values $x_1, \ldots, x_m$ of cross ratio parameters. Let

$$W_j = A_1 A_2 \cdots A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad j = 1, \ldots, m,$$

where $A_i$ is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & x_i \end{pmatrix}$ associated to $e_i$. Then, it was verified in [9] that for each vertex $v$ of $\tau$, we have

$$W_v = A_1 A_2 \cdots A_m = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.1}$$

and

$$\begin{cases} a_j,\ c_j < 0,\ b_j,\ d_j > 0 \ \text{ for } \ 1 \leq j \leq m-1 \\ \text{except for } \ a_1 = d_{m-1} = 0. \end{cases} \tag{4.2}$$

The first condition comes from the fact that the chain of circles surrounding the circle corresponding to $v$ closes up. The second condition excludes over-winding, that is, it eliminates the case where the chain surrounds the central circle more than once. Notice here that the associated matrices are in $\mathrm{SL}(2,\mathbb{R})$ and not in $\mathrm{PSL}(2,\mathbb{R})$, so that the inequalities of (4.2) make sense.

On the other hand, given a real valued map $\mathbf{x}$ of $E_\tau$ satisfying (4.1) and (4.2) for each vertex of $\tau$, it is relatively routine to construct a pair $(S,P)$ of a projective Riemann surface $S$ and a circle packing $P$ on $S$ so that its cross ratio parameter is $\mathbf{x}$ (see [9] for details). Thus set

$$\mathcal{C}_\tau = \{\mathbf{x} : E_\tau \to \mathbb{R} \,|\, \mathbf{x} \text{ satisfies (4.1) and (4.2) for each vertex}\},$$

and call it the *cross ratio parameter space*.

**Remark 4.1.** In §3.3, we have defined $\mathcal{C}_\tau$ as the moduli space of circle configurations on $\widehat{\mathbb{C}}$ with nerve isotopic to $\tau$ up to projective equivalence endowed with a natural topology. When $\tau$ defines a cell decomposition by triangles, the space $\mathcal{C}_\tau$ is a point. A nontrivial moduli space appears only when the decomposition contains a quadrilateral cell. Here we use the same notation since $\mathcal{C}_\tau$ is naturally identified with the moduli space of circle packings on projective Riemann surfaces of genus $g \geq 1$ controled by $\tau$. It will be nontrivial even if $\tau$ defines a cell decomposition by only triangles.

Since the condition (4.1) gives a set of polynomial equations for the $x_i$'s and (4.2) are polynomial inequalities in the $x_i$'s, A moduli space $\mathcal{C}_\tau$ is a semi-algebraic set by definition, and we define the topology on $\mathcal{C}_\tau$ to be the one induced by the tautological inclusion $\imath : \mathcal{C}_\tau \to \mathbb{R}^{E_\tau}$. It turns out that this naive construction gives a correct parameterization of the moduli space of pairs $(S,P)$ where $S$ is a projective Riemann surface and $P$ is a circle packing on S with nerve $\tau$.

**Lemma 4.2.** *If $g \geq 1$, and a simple graph $\tau$ on $\Sigma_g$ defines a triangulation in the universal cover, then we have the following :*

(1) (Lemma 2.17 in [9]) *A moduli space $\mathcal{C}_\tau$ corresponds bijectively to the set of all pairs $(S, P)$ where $S$ is a projective Riemann surface and $P$ is a circle packing on $S$ with nerve $\tau$, up to marked projective equivalence.*

(2) (Lemma 3.2 in [10]) *The tautological inclusion $\imath : \mathcal{C}_\tau \to \mathbb{R}^{E_\tau}$ is proper.*

In view of the above results, $\mathcal{C}_\tau$ is naturally identified with the moduli space of all pairs $(S, P)$ with nerve $\tau$. The study of the moduli space then reduces to the study of its semi-algebraic representative $\mathcal{C}_\tau$.

## 4.2 Thurston Coordinates

To each pair $(S, P)$ in $\mathcal{C}_\tau$, assign only its first component and we obtain the *forgetting map*

$$f : \mathcal{C}_\tau \longrightarrow \mathcal{P}_g.$$

The image $f(\mathcal{C}_\tau)$ consists of all projective Riemann surfaces which admit a circle packing with nerve $\tau$. The projective rigidity implies that $f(\mathcal{C}_\tau)$ intersects $s(\mathcal{T}_g)$ only at $f(\{\text{KAT}\})$ and furthermore, the rigidity of the circle packing on $f(\{\text{KAT}\})$ means that the inverse image of this point under $f$ consists of exactly one point. We discuss here the description of $f(\mathcal{C}_\tau)$ with respect to Thurston coordinates of $\mathcal{P}_g$ which we will describe shortly.

We assume that the surface has genus $g \geq 2$ in this subsection. A measured lamination is defined to be a closed subset on $\Sigma_g$ locally homeomorphic to a product of a totally disconnected subset of the interval with an interval, together with a transverse measure. Moreover, we restrict ourselves to the case when every leaf is homotopic to a geodesic with respect to some (and hence any) hyperbolic metric on $\Sigma_g$. A noncontractible simple closed curve on $\Sigma_g$ with counting measure for transverse arcs is an elementary, but important and fundamental example of a measured lamination. The space of isotopy classes of measured laminations on $\Sigma_g$ ($g \geq 2$) with weak * topology on measures will be denoted by $\mathcal{ML}_g$. The set of weighted homotopically nontrivial simple closed curves is dense in $\mathcal{ML}_g$. Also $\mathcal{ML}_g$ is known to be homeomorphic to $\mathbb{R}^{6g-6}$. See [17, 18] for details.

Although a measured lamination is a topological concept, once we put a hyperbolic metric on $\Sigma_g$, its support is canonically realized as a disjoint union of simple geodesics which forms a closed subset on the surface. Such a lamination is called a geodesic lamination with transverse measure.

Thurston has assigned to each projective Riemann surface a hyperbolic surface with a measured geodesic lamination. Following [7], we briefly review his idea. Start with a projective Riemann surface $S$ which is not a hyperbolic surface. Consider the set of maximal disks in the universal cover $\widetilde{S}$. Each maximal disk is naturally endowed with the hyperbolic metric, the boundary of each disk intersects the ideal boundary of $\widetilde{S}$ in two or more points and we can take the convex hull of these ideal boundary points. It can be shown that this gives a stratification of $\widetilde{S}$ by ideal polygons, and ideal bigons foliated by "parallel lines" joining the two ideal vertices of the bigons. The polygonal parts support a canonical hyperbolic metric. Collapsing each bigon foliated by parallel lines in $\widetilde{S}$ to a line and taking the quotient of the result by the action of the fundamental group, we obtain a hyperbolic surface $H$. This defines a hyperbolization map

$$\alpha : \mathcal{P}_g \longrightarrow s(\mathcal{T}_g) \subset \mathcal{P}_g.$$

Also the stratification defines a geodesic lamination $\lambda$ on $H$ by taking the union of collapsed lines. Moreover, identifying $\widehat{\mathbb{C}}$ with the boundary of the 3-dimensional hyperbolic space $\mathbb{H}^3$, and using the convex hull of the ideal points of the maximal disk not on the disk itself but in $\mathbb{H}^3$, we can assign a transverse bending measure supported on $\lambda$. This defines a pleating map

$$\beta : \mathcal{P}_g \longrightarrow \mathcal{ML}_g.$$

**Theorem 4.3** (Thurston, see [7]). *The product of these maps*

$$(\alpha, \beta) : \mathcal{P}_g \longrightarrow s(\mathcal{T}_g) \times \mathcal{ML}_g, \tag{4.3}$$

*is a homeomorphism.*

We call the parameterization of $\mathcal{P}_g$ by the target of (4.3) Thurston coordinates. It is known that the restriction of $\pi : \mathcal{P}_g \to \mathcal{T}_g$ to any slice $s(\mathcal{T}) \times \{*\}$ by the first factor is proper by Tanigawa [16], and then locally injective by Scannell and Wolf [13]. In particular, it is a diffeomorphism. Also Dumas and Wolf proved that the same is true for the slice $\{*\} \times \mathcal{ML}_g$ by the second factor in [6].

On the other hand,

**Lemma 4.4** (Lemma 4.1 in [10]). *If $g \geq 2$, then the composition $\beta \circ f : \mathcal{C}_\tau \to \mathcal{ML}_g$ of a forgetting map $f : \mathcal{C}_\tau \to \mathcal{P}_g$ with the pleating map $\beta : \mathcal{P}_g \to \mathcal{ML}_g$ has bounded image.*

This is a property for projective Riemann surfaces admitting a circle packing dominated by a single graph $\tau$, and is proved by observing how the developed image of a projective Riemann surface is controlled by the combinatorial data of $\tau$.

## 4.3 Speculation

To expect a deeper understanding of the moduli space $\mathcal{C}_\tau$ and its image in $\mathcal{P}_g$ under $f$, we formulate a conjecture which relates the moduli space with Teichmüller space.

**Conjecture 4.5.** The composition $\pi \circ f : \mathcal{C}_\tau \to \mathcal{T}_g$ of the forgetting map $f : \mathcal{C}_g \to \mathcal{P}_g$ with the projection $\pi : \mathcal{P}_g \to \mathcal{T}_g$ is a homeomorphism.

The motivation goes back to the result of Mizushima in [12] which we discuss in the next subsection. Here are some expected implications of the affirmative solution to Conjecture 4.5, which have been verified in certain special cases.

(1) (Topology of $\mathcal{C}_\tau$) The moduli space $\mathcal{C}_\tau$ would be homeomorphic to the euclidean space of dimension 2 or $6g - 6$ according to whether $g = 1$ or $g \geq 2$.

(2) (Rigidity for circle packings) The forgetting map $f : \mathcal{C}_\tau \to \mathcal{P}_g$ would be injective. Thus the rigidity of circle packings holds for all projective Riemann surfaces in $f(\mathcal{C}_\tau)$, that is, each projective Riemann surface $S$ admits at most one circle packing with nerve $\tau$ up to projective automorphisms isotopic to the identity.

(3) (New section to $\pi$) The image $f(\mathcal{C}_\tau)$ of the forgetting map would define a new natural section or a slice to $\pi : \mathcal{P}_g \to \mathcal{T}_g$. It means for example that for each biholomorphic class of a Riemann surface, there exists a unique projective Riemann surface which admits a circle packing with nerve $\tau$.

## 4.4 Evidence

**4.4.1** Consider the moduli space $\mathcal{C}_\tau$ of circle packings by one circle on $\Sigma_1$. The nerve $\tau$ in this case consists of one vertex $v$ and 3 edges $e_1$, $e_2$ and $e_3$ with cross ratios $x > 0$, $y > 0$ and $z > 0$ respectively and associated matrices $X = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}$ and $Z = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$. The word associated to the vertex is given by $W = XYZXYZ$, and $XYZXYZ = I$ implies

$$xyz = x + y + z$$

Note that three equations derived from the matrix identity reduces to just one equation in this case. By an easy computation, we can see that the cross ratio parameter space $\mathcal{C}_\tau$ is given by

$$\mathcal{C}_\tau = \{(x, y, z) \in \mathbb{R}^3 |\ xyz = x + y + z,\ x, y, z > 0\},$$

which is homeomorphic to a convex domain in the $xy$-plane,

$$\{(x, y) \in \mathbb{R}^2 \mid xy - 1 > 0,\ x, y > 0\},$$

by the projection.

Mizushima studied the moduli space of complex affine structures on the torus in [12]. When we translate his result to our language, his moduli space doubly branched covers to the moduli space in this setting and provides an affirmative solution to Conjecture 4.5 for this very special case.

**Theorem 4.6** (Mizushima [12]). *If a simple graph $\tau$ on $\Sigma_1$ has only one vertex, then the composition $\pi \circ f : \mathcal{C}_\tau \to \mathcal{T}_1$ of the forgetting map $f : \mathcal{C}_\tau \to \mathcal{P}_1$ with the projection $\pi : \mathcal{P}_1 \to \mathcal{T}_1$ is a homeomoprhism.*

**4.4.2** The argument developed by Brooks, described in §3.3, is extendable to projective Riemann surfaces such that a developing map extends to an embedding of the closure of the universal cover. Such a surface is here called *strongly uniformizable*. Let $\tau$ be a simple graph on $\Sigma_g$ which defines a cell decomposition with only triangular and quadrilateral cells, $S$ a strongly uniformizable surface, $\Delta$ the image of a holonomy representation in $\mathrm{PGL}(2, \mathbb{C})$ and $C$ a circle configuration on $S$ whose nerve is isotopic to $\tau$. In this case, the universal cover $\widetilde{S}$ is embedded in $\widehat{\mathbb{C}}$, but the boundary $\partial\widetilde{S}$ would not be a round circle and is a quasi circle on $\widehat{\mathbb{C}}$ in general. Adding to $\Delta$ only reflections about members of $\widetilde{C}$ without $C_0$ in §3.3, we get a Kleinian group $\Gamma$. Then the quasi-conformal deformation theory tells us that $\mathcal{QC}(\Gamma)$ is homeomorphic to the product of the Teichmüller spaces of the quadrilateral compementary regions of $C \subset S$ and $\mathcal{T}_g$ which corresponds to the outside region of the closure of $\widetilde{S}$ in $\widehat{\mathbb{C}}$.

Suppose that $C$ is a circle packing, then $\mathcal{QC}(\Gamma)$ is homeomorphic to $\mathcal{T}_g$. This shows that there is a family of projective Riemann surfaces parametrized by $\mathcal{T}_g$ which admit a circle packing with nerve isotopic to $\tau$, and in fact we have an embedding

$$\mathcal{T}_g \cong \mathcal{QC}(\Gamma) \longrightarrow \mathcal{QC}(\Delta) \cong \mathcal{T}_g \times \mathcal{T}_g \subset \mathcal{P}_g$$

This observation establishes the local structure of $\mathcal{C}_\tau$ at the KAT solution for $g \geq 2$.

When $g = 1$, the hyperbolic Dehn surgery theory developed by Thurston [17] plays a similar role with the quasi-deformation theory at least locally at the KAT solution. Thus one may establish

**Theorem 4.7** (Theorem 1 in [9]). *There is a neighborhood $U$ of the* KAT *pair in $\mathcal{C}_\tau$ such that*

(1) *$U$ is homeomorphic to the euclidean space of dimension* 2 *or* $6g - 6$ *according to whether $g = 1$ or $g \geq 2$,*

(2) *the restriction of $f$ to $U$ is injective.*

**4.4.3** The restriction for $\tau$ to have only one vertex as in Mizushima's setting simplifies the situation even for the case $g \geq 2$. In fact, $\mathcal{C}_\tau$ is defined by just one matrix equation and the set of inequalities corresponding to (4.1) and (4.2) respectively. This rather simple setting enable us to prove for example,

**Theorem 4.8.** *If $\tau$ has one vertex and $g \geq 2$, then*
(1) (Theorem 2 in [9]) $\mathcal{C}_\tau$ *is homeomorphic to* $\mathbb{R}^{6g-6}$.
(2) (Lemma 5.1 in [9]) $f : \mathcal{C}_\tau \to \mathcal{P}_g$ *is injective.*
(3) (Theorem 1.1 in [10]) $\pi \circ f : \mathcal{C}_\tau \to \mathcal{T}_g$ *is proper.*

Theorem 4.8 comes fairly close to affirmatively answering to Conjecture 4.5 for the one circle packing case. What is missing is a proof that $p$ restricted to $f(\mathcal{C}_\tau)$ is locally injective.

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