

CIRCLE PACKINGS ON SURFACES WITH PROJECTIVE STRUCTURES AND UNIFORMIZATION

SADAYOSHI KOJIMA, SHIGERU MIZUSHIMA, AND SER PEOW TAN

ABSTRACT. Let Σ_g be a closed orientable surface of genus $g \geq 2$ and τ a graph on Σ_g with one vertex which lifts to a triangulation of the universal cover. We have shown that the cross ratio parameter space \mathcal{C}_τ associated with τ , which can be identified with the set of all pairs of a projective structure and a circle packing on it with nerve isotopic to τ , is homeomorphic to \mathbb{R}^{6g-6} , and moreover that the forgetting map of \mathcal{C}_τ to the space of projective structures is injective. In this paper, we show that the composition of the forgetting map with the uniformization from \mathcal{C}_τ to the Teichmüller space \mathcal{T}_g is proper.

1. INTRODUCTION

Let Σ_g be a closed orientable surface of genus $g \geq 2$ and τ a graph on Σ_g which lifts to a triangulation of the universal cover $\tilde{\Sigma}_g$. In [7], we initiated the study of circle packings on Σ_g with projective structures with combinatorics controlled by τ . Since we discuss projective structures which are complex structures in particular, we orient Σ_g throughout this paper so that the orientation is the same as the one coming from the complex structure.

As the circle/disk is a fundamental geometric notion for projective structures, it makes sense to ask which surfaces admit a circle packing with fixed nerve isotopic to τ , and furthermore, if the circle packings carried on these surfaces are rigid. The main results of [7] answer these two questions when τ has exactly one vertex. In this case, the cross ratio parameter space \mathcal{C}_τ associated with a graph τ on Σ_g , which can be identified with the set of all pairs of a projective structure and a

2000 *Mathematics Subject Classification*. Primary 52C15; Secondary 30F99, 57M50.

Key words and phrases. circle packing, projective structure, uniformization, Teichmüller space.

The third author gratefully acknowledge support from the National University of Singapore academic research grant R-146-000-031-112 and the hospitality of the Department of Mathematical and Computing Sciences, Tokyo Institute of Technology.

circle packing on it with nerve isotopic to τ , is homeomorphic to \mathbb{R}^{6g-6} . Furthermore, the forgetting map,

$$f : \mathcal{C}_\tau \rightarrow \mathcal{P}_g,$$

of \mathcal{C}_τ to the space \mathcal{P}_g of projective structures on Σ_g which forgets the packing is injective. Namely, the packings are in fact rigid.

On the other hand, any projective structure on Σ_g has a canonical underlying complex structure. Thus assigning the underlying complex structure to each projective structure, we obtain the uniformization map

$$u : \mathcal{P}_g \rightarrow \mathcal{T}_g,$$

of \mathcal{P}_g to the Teichmüller space \mathcal{T}_g (thought of here as the space of complex structures). By taking the Schwarzian derivative of the developing map, a projective structure can be identified with a holomorphic quadratic differential over the underlying Riemann surface, so the uniformization map is a complex vector bundle of rank $3g - 3$ over \mathcal{T}_g .

For the genus one case, when τ has one vertex, it was shown by Mizushima in [8], with slightly different language, that the composition of the forgetting map with the uniformization map is a homeomorphism. In [7], we conjectured that this holds in general, regardless of the genus $g (\geq 1)$ and the graph τ . The purpose of this paper is to take the first step towards this conjecture by proving the following properness theorem for τ with one vertex:

Theorem 1.1. *Let τ be a graph on Σ_g ($g \geq 2$) with one vertex which lifts to a triangulation of $\tilde{\Sigma}_g$ and \mathcal{C}_τ the cross ratio parameter space associated with τ . Then the composition $u \circ f : \mathcal{C}_\tau \rightarrow \mathcal{T}_g$ of the forgetting map with the uniformization map is proper.*

To complete the proof of the conjecture for such graphs τ , since \mathcal{C}_τ in this case was shown to be homeomorphic to \mathbb{R}^{6g-6} , it suffices to show that the map is locally injective. This sort of question for the grafting map based on Tanigawa's properness theorem in [11] was settled by Scannell and Wolf in [10]. See also Faltings [3] and McMullen [9] for earlier proofs of special cases. However, it is not clear if the proofs in the above cited papers can be extended to our setting.

The rest of this paper is organized as follows. In §2, we set the notations, and results required to prove the main theorem. In §3, we recall the definition of the cross ratio parameter space from [7] and show that it is properly embedded in the euclidean space. In §4, following the exposition in [5], we briefly review the Thurston coordinates of \mathcal{P}_g in terms of hyperbolic structures and measured laminations, and show that the projected image of $f(\mathcal{C}_\tau)$ to the space of measured laminations

on Σ_g is bounded. The results up to that section are valid for any graph τ . In §5, under the assumption that τ has exactly one vertex, we show that the holonomy map from $f(\mathcal{C}_\tau)$ to the algebro-geometric quotient of the space of representations of $\pi_1(\Sigma_g)$ in $\mathrm{PSL}_2(\mathbb{C})$ up to conjugacy is proper, and deduce from this that the projection of $f(\mathcal{C}_\tau)$ to the space of the hyperbolic structures in the Thurston coordinates is also proper. Finally, in §6, we complete the proof of the theorem using Tanigawa's inequality in [11].

2. PRELIMINARIES

We first set the notation and recall results from [5, 7] to be used in the rest of the paper. Since g and τ are fixed throughout this paper, we will drop the suffices where possible to simplify the notation. Let $\Sigma_g = \Sigma$ be a closed oriented surface of genus $g \geq 2$.

Attached to Σ , we have the following spaces.

- The Teichmüller space $\mathcal{T}_g = \mathcal{T}$ which parameterizes either the hyperbolic or the complex structures on Σ , depending on the point of view.
- The measured lamination space $\mathcal{ML}_g = \mathcal{ML}$ which is the completion of the space of weighted simple closed curves on Σ . It is known to be homeomorphic to \mathbb{R}^{6g-6} .
- The space $\mathcal{P}_g = \mathcal{P}$ of projective structures on Σ , namely a geometric structure locally modeled on the Riemann sphere with transition functions in $\mathrm{PSL}_2(\mathbb{C})$.
- The algebro-geometric quotient $\mathcal{X}_g = \mathcal{X}$ of the space of representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}_2(\mathbb{C})$ up to conjugacy. Taking the holonomy representation gives a map,

$$\mathrm{hol} : \mathcal{P} \mapsto \mathcal{X},$$

which is a local homeomorphism, see [4].

Furthermore, we have the following spaces associated to Σ and a graph τ triangulating Σ by the study in [7].

- The cross ratio parameter space $\mathcal{C}_\tau = \mathcal{C}$, which turns out to be a semi-algebraic set in \mathbb{R}^E , where E is the set of edges of τ . An element $\mathbf{c} \in \mathcal{C} \subset \mathbb{R}^E$ is called a cross ratio parameter and will be reviewed in the next section. It was shown in [7] that each cross ratio parameter determines a surface $S \in \mathcal{P}$ with a projective structure together with a circle packing P on S with nerve isotopic to τ . Conversely, each pair (S, P) gives a cross ratio parameter \mathbf{c} in \mathcal{C} .

- The image $\mathcal{I} = f(\mathcal{C})$ of \mathcal{C} by the forgetting map $f : \mathcal{C} \rightarrow \mathcal{P}$, which is the set of all projective structures on Σ which admit a circle packing with nerve isotopic to τ . When τ has only one vertex, it was shown in [7] that f is a homeomorphism onto the image \mathcal{I} and hence f identifies \mathcal{C} with \mathcal{I} under this condition.

Here is another view of \mathcal{P} .

- The Thurston coordinates of \mathcal{P} ,

$$\mathcal{P} \cong \mathcal{T} \times \mathcal{ML},$$

which use the pleated hyperbolic surface corresponding to a projective structure. The first factor parameterizes the hyperbolic surface and the second factor parameterizes the measured lamination coming from the bending. The coordinates are reviewed briefly in §4.

3. THE CROSS RATIO PARAMETER SPACE \mathcal{C}

We first recall briefly how the cross ratio parameter space \mathcal{C} in [7] was defined. Suppose that S is a surface with projective structure which lies in $\mathcal{I} \subset \mathcal{P}$ and P is a circle packing on S with nerve τ . To each edge e of τ is associated a configuration of four circles in the developed image surrounding a preimage \tilde{e} of e , see Figure 1(a). In [7], we defined a cross ratio of an edge e by taking the imaginary part of the cross ratio of four contact points $(p_{14}, p_{23}, p_{12}, p_{13})$ of the configuration chosen as in Figure 1(b) with orientation convention (for the definition of the cross ratio of four ordered points, see [1]). It is the modulus of the rectangle obtained by normalizing the configuration by moving the contact point p_{13} to ∞ , see Figure 2.

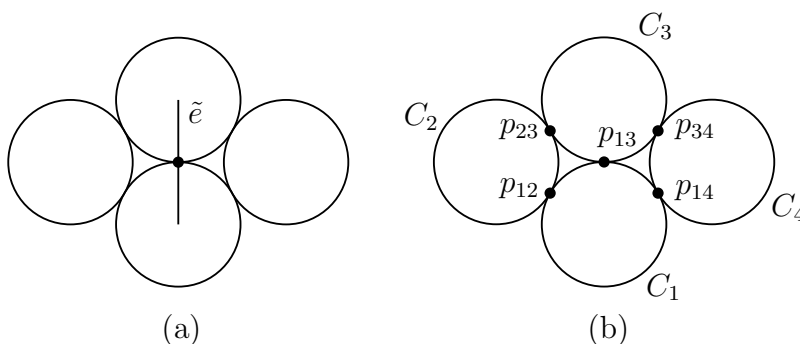


FIGURE 1

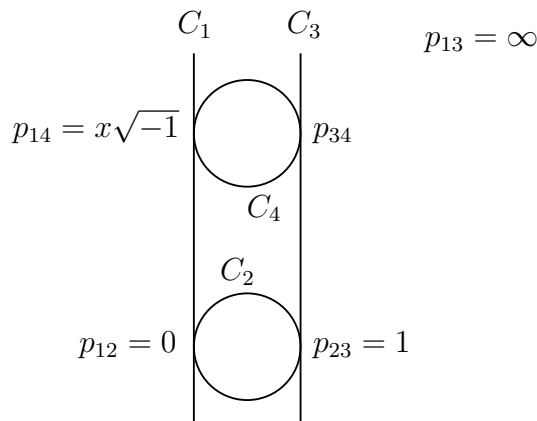


FIGURE 2

Because the developing map is a local homeomorphism, the cross ratios of the edges must satisfy certain conditions, best expressed in terms of the associated matrices. If e is an edge of τ with a cross ratio x , we associate to e the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$. Now, if v is a vertex of τ with valence m , we read off the edges e_1, \dots, e_m incident to v in a clockwise direction to obtain a sequence of cross ratios x_1, \dots, x_m associated to v . Let

$$W_j = A_1 A_2 \cdots A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad j = 1, \dots, m$$

where A_i is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & x_i \end{pmatrix}$ associated to e_i . Then, for each vertex v of τ , we have

$$W_v = A_1 A_2 \cdots A_m = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1)$$

and

$$\begin{cases} a_j, c_j < 0, b_j, d_j > 0 \text{ for } 1 \leq j \leq m-1 \\ \text{except for } a_1 = d_{m-1} = 0. \end{cases} \quad (2)$$

Roughly, the first condition ensures the consistency of the chain of circles surrounding the circle corresponding to v , see Figure 3, and the second condition eliminates the case where the chain surrounds the central circle more than once. We note that it does not matter which edge we start out with in the above.

Conversely, if an assignment to the edges of τ ,

$$\mathbf{c} : E \longrightarrow \mathbb{R},$$

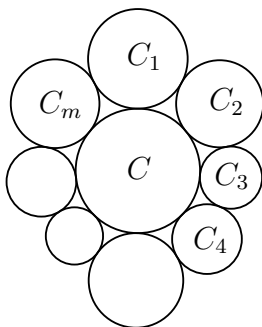


FIGURE 3

satisfies conditions (1) and (2) for each vertex, the map is a cross ratio parameter of some packing P on a surface $S \in \mathcal{I}$, where both S and P are determined by \mathbf{c} (main lemma of [7]). Hence, the set

$$\mathcal{C} = \{\mathbf{c} : E \rightarrow \mathbb{R} \mid \mathbf{c} \text{ satisfies (1) and (2) for each vertex}\},$$

called the *cross ratio parameter space*, can be identified with the space of pairs (S, P) of a surface S with projective structure and a circle packing P on S , and \mathbf{c} parameterizes the space of such pairs.

By the definition, we see that \mathcal{C} is a semi-algebraic set in \mathbb{R}^E defined by equations coming from (1) and inequalities coming from (2) for each vertex v . We first show that the strict inequalities of (2) can be replaced by non-strict inequalities, namely, for each vertex v and with the same notation as before, consider the set of conditions

$$\begin{cases} a_j, c_j \leq 0, b_j, d_j \geq 0 \text{ for } 1 \leq j \leq m-1 \\ \text{except for } a_1 = d_{m-1} = 0 \end{cases} \quad (3)$$

Lemma 3.1. *The conditions (1),(2) \iff (1),(3).*

Proof. It is sufficient to prove (1), (3) \implies (2). We have the identity

$$\begin{aligned} \begin{pmatrix} a_{j+1} & b_{j+1} \\ c_{j+1} & d_{j+1} \end{pmatrix} &= \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x_{j+1} \end{pmatrix} \\ &= \begin{pmatrix} -b_j & a_j + b_j x_{j+1} \\ -d_j & c_j + d_j x_{j+1} \end{pmatrix}. \end{aligned} \quad (4)$$

Now $a_2 = -b_1 = -1 < 0$, so by induction, using (3) and (4), $a_{j+1} = -b_j < 0$ for $j = 1, \dots, m-1$ since

$$a_2 < 0 \implies b_2 > 0 \implies a_3 < 0 \implies \dots \implies b_{m-1} > 0 \implies a_m < 0.$$

Similarly, since $c_1 = -1 < 0$, by induction, using (3) and (4), $c_{j+1} = -d_j < 0$ for $j = 1, \dots, m-2$ since

$$c_1 < 0 \implies d_1 > 0 \implies c_2 < 0 \implies \dots \implies d_{m-2} > 0 \implies c_{m-1} < 0.$$

□

This lemma tells us that the condition (2) does not divide a connected component of the algebraic set determined by (1). It just chooses appropriate connected components. With this, we can easily prove

Lemma 3.2. *The inclusion map of \mathcal{C} into \mathbb{R}^E is proper, where E is the set of edges of τ .*

Remark. The above result holds for a general τ with no restriction on the number of vertices.

Proof. Let $\{\mathbf{c}_n\}$ be a sequence of points in the intersection of a compact set in \mathbb{R}^E with \mathcal{C} . Then, there exists a convergent subsequence. Let \mathbf{c}_∞ denote the limit of the subsequence. Each \mathbf{c}_n satisfies the conditions (1) and (2). Each entry of the product of the matrices in the conditions (1) and (2) is a polynomial in terms of x_i 's and thus is continuous. Hence, the limit \mathbf{c}_∞ satisfies the conditions (1) and (3), and also lies in \mathcal{C} by lemma 3.1. □

4. IMAGE IN \mathcal{ML} IS BOUNDED

Suppose $S \in \mathcal{P}$ is a surface with projective structure. Recall from [5] that associated to the developed image $dev(\tilde{S})$ is a collection of maximal disks $\{D_i\}$ on \tilde{S} . To each maximal disk, there is a set of at least two ideal boundary points which lie in the closure of D_i but not in \tilde{S} . Take the convex hull $V_i \subset D_i$ of the ideal boundary points with respect to the hyperbolic metric on the interior of D_i . Then for two distinct maximal disks D_i and D_j , $V_i \cap V_j = \emptyset$ and the collection $\{V_i\}$ gives a stratification of \tilde{S} equivariant under the action of the fundamental group. Note that each V_i is either an ideal polygon or a line, except in the trivial case where S is a hyperbolic structure, in which case there is only one maximal disk. Also every point z in \tilde{S} is contained in a unique convex hull V_i . We call the maximal disk D_i containing V_i a supporting maximal disk of z .

The polygonal parts of the stratification $\{V_i\}$ support the hyperbolic metric. Collapsing parallel lines in $\{V_i\}$ to a line and quotienting the resultant by the action of the fundamental group, we obtain a map of S to a hyperbolic surface H . This collapsing construction gives a map

$$\pi : \mathcal{P} \rightarrow \mathcal{T}$$

where \mathcal{T} is regarded here as the space of hyperbolic structures on Σ .

The stratification $\{V_i\}$ also defines a geodesic lamination λ on H by taking the complement of the interior of polygonal parts. Moreover,

using the convex hull of the ideal points of the maximal disk not in the disk but in the 3-dimensional hyperbolic space \mathbb{H}^3 , we can assign a transverse bending measure supported on λ . Hence we also obtain a map

$$\beta : \mathcal{P} \rightarrow \mathcal{ML}.$$

Thurston showed that the pair (π, β) of these maps is a homeomorphism between \mathcal{P} and $\mathcal{T} \times \mathcal{ML}$.

Lemma 4.1. $\beta(\mathcal{I})$ is bounded in \mathcal{ML} .

Remark. The above result holds for a general τ with no restriction on the number of vertices.

Proof. Let S be a surface in $\mathcal{I} = f(\mathcal{C}) \subset \mathcal{P}$, P a circle packing on S with nerve τ , and $H = \pi(S)$, $\lambda = \beta(S)$. The measured lamination λ can be pulled back to a lamination with transverse measure by blowing up the atomic leaves on H to parallel leaves in S with stretched transverse measure in a canonical way. Hence we regard S as a surface with this measured lamination μ . To see boundedness of $\beta(\mathcal{I})$, it is sufficient to show that the measure along each edge of τ is uniformly bounded since τ generates the fundamental group of Σ and μ collapses to λ .

We can choose a reference point for each circle in P to represent the vertex v of τ such that the supporting maximal disk of the point representing v contains the circle and take their preimage in \tilde{P} to get equivariant reference vertices. Let C_1 and C_2 be contact circles in \tilde{P} , and D_i the supporting maximal disk of the reference point v_i of C_i . D_i contains C_i for both $i = 1, 2$, so these form a π roof over the pleated hyperbolic surface in \mathbb{H}^3 locally corresponding to $dev(\tilde{S})$, see Figure 4. So the total transverse measure along a path \tilde{e} between v_1 and v_2 contained in $D_1 \cup D_2$ is bounded above by π .

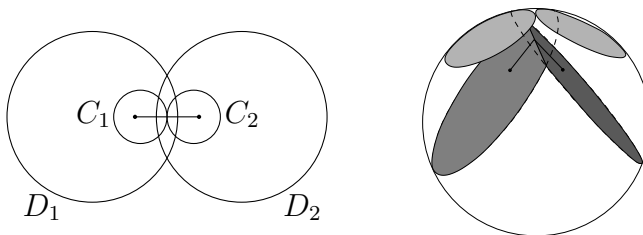


FIGURE 4

□

5. *hol* AND π ON IMAGE ARE PROPER

In this section, we restrict ourselves to the case where τ has one vertex. We first show

Lemma 5.1. *Suppose that τ has only one vertex. Then there is a finite subset F of the fundamental group $\pi_1(\Sigma)$ such that if $\{\mathbf{c}_n\}$ is a sequence of points in \mathcal{C} which escapes away from the compact sets, then there is an element $g \in F$ and a subsequence of $\{\mathbf{c}_n\}$ for which $|\text{tr}(\rho_n(g))| \rightarrow \infty$, where $\rho_n = \text{hol}(f(\mathbf{c}_n))$. In particular, the composition $\text{hol} \circ f : \mathcal{C} \rightarrow \mathcal{X}$ of the forgetting map with $\text{hol} : \mathcal{I} \rightarrow \mathcal{X}$ is proper.*

Remark. Although the image of holonomy lies in $\text{PSL}_2(\mathbb{C})$ and the trace makes sense only up to signs, its absolute value is well-defined.

Proof. Since τ has one vertex, each edge e_i of τ starts and ends at the same vertex v , so corresponds to a pair of elements $g_i^{\pm 1} \in \pi_1(\Sigma)$, and the set $\{g_i^{\pm 1}\}$ forms a generating set for $\pi_1(\Sigma)$. F is defined to be the set of all words of length 2 in this generating set.

Passing to a subsequence, we may assume that the cross ratio $\mathbf{c}_n(e)$ of some fixed edge e of τ approaches ∞ as $n \rightarrow \infty$. For each $S_n = f(\mathbf{c}_n)$, consider the developed image of \tilde{S}_n and in particular the configuration of 6 circles C_1, \dots, C_6 in \tilde{P}_n with corresponding vertices v_1, \dots, v_6 of $\tilde{\tau}$ as given in Figure 5a. Here, the edge $e_3 = v_1v_3$ is the one whose cross ratio approaches ∞ as $n \rightarrow \infty$. Note that the configuration of C_j and v_j depends on n .

To each n , we may normalize the developed image so that it is given by Figure 5b, where the tangency point p_{13} between C_1 and C_3 is $\sqrt{-1}$. The concatenation of the two directed edges $e_2 = v_2v_1$ and $e_4 = v_1v_4$ which are the neighboring edges of e_3 about the vertex v_1 corresponds to an element $g = g_4g_2^{-1} \in F$. Its holonomy image $\varphi_n := \rho_n(g) = \rho_n(g_4g_2^{-1})$ is an element of $\text{PSL}_2(\mathbb{C})$ mapping C_2 to C_4 . Note that in the normalized picture, the radius of C_4 approaches zero as $n \rightarrow \infty$ since $\mathbf{c}_n(e_3) \rightarrow \infty$. Hence we can represent the images of 0, 1 and ∞ , all lying on C_2 , under φ_n by $\sqrt{-1} + \varepsilon_1$, $\sqrt{-1} + \varepsilon_2$ and $\sqrt{-1} + \varepsilon_3$ respectively, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \rightarrow 0$ as $n \rightarrow \infty$. Letting $K_n = \frac{\varepsilon_2 - \varepsilon_3}{\varepsilon_2 - \varepsilon_1}$, φ_n^{-1} is then described by

$$\varphi_n^{-1} : z \mapsto K_n \frac{z - (\sqrt{-1} + \varepsilon_1)}{z - (\sqrt{-1} + \varepsilon_3)}, \text{ and}$$

$$M_n = \frac{1}{\sqrt{K_n(\varepsilon_1 - \varepsilon_3)}} \begin{pmatrix} K_n & -K_n(\sqrt{-1} + \varepsilon_1) \\ 1 & -(\sqrt{-1} + \varepsilon_3) \end{pmatrix} \in \text{SL}_2(\mathbb{C})$$

is a matrix representative of φ_n^{-1} . We have

$$\operatorname{tr}(M_n) = \frac{1}{\sqrt{\varepsilon_1 - \varepsilon_3}} \left(\sqrt{K_n} - \frac{1}{\sqrt{K_n}} (\sqrt{-1} + \varepsilon_3) \right)$$

and $|\operatorname{tr}(\varphi_n)| = |\operatorname{tr}(\varphi_n^{-1})| = |\operatorname{tr}(M_n)|$. If $|\operatorname{tr}(M_n)| \rightarrow \infty$, we are done, otherwise, we must have $(\sqrt{K_n} - \frac{1}{\sqrt{K_n}}(\sqrt{-1} + \varepsilon_3)) \rightarrow 0$ since $(\varepsilon_1 - \varepsilon_3) \rightarrow 0$ as $n \rightarrow \infty$. This implies that K_n approaches $\sqrt{-1}$ since $\varepsilon_3 \rightarrow 0$ too. Geometrically, this means that the angle formed by $\sqrt{-1} + \varepsilon_1$, $\sqrt{-1} + \varepsilon_2$ and $\sqrt{-1} + \varepsilon_3$ approaches $\pi/2$, and hence $\sqrt{-1} + \varepsilon_1$ and $\sqrt{-1} + \varepsilon_3$ approach diametrically opposite positions on the circle C_4 . In other words, the arc $\alpha_1 = \varphi_n([-\infty, 0])$ occupies half of the circle C_4 in the limit.

On the other hand, the inferior arc α_2 on C_4 connecting p_{14} and p_{34} also occupies half of C_4 in the limit since the radius of C_4 , which we denote by $\operatorname{rad}(C_4)$, approach 0, where p_{jk} is the tangency point between C_j and C_k .

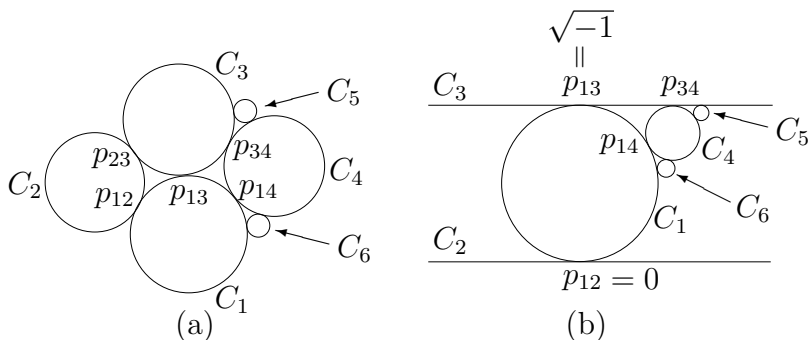


FIGURE 5

Claim: α_1 and α_2 are the images under φ_n of two distinct, non-adjacent segments of C_2 and thus are distinct non-adjacent segments on C_4 .

Proof of Claim. It suffices to show that the image of $[-\infty, 0]$ on C_1 by $\rho_n^{-1}(g_2)$ and the image of the arc α_2 by $\rho_n^{-1}(g_4)$ are non-adjacent segments of C_1 . The pair of edges (e_2, e_3) are either part of an intersecting or non-intersecting triple of edges, since they are adjacent (see [7], §4). Similarly, the pair of edges (e_3, e_4) are also part of an intersecting or non-intersecting triple. For all possibilities, $\rho_n^{-1}(g_2)[-\infty, 0]$ and $\rho_n^{-1}(g_4)(\alpha_2)$ are non-adjacent segments of C_1 , see Figure 6 (a), (b) and (c). \square

From the claim, the subtended angle between p_{45} and p_{34} on C_4 in Figure 5 (b) approaches zero, as does the angle between p_{46} and p_{14} .

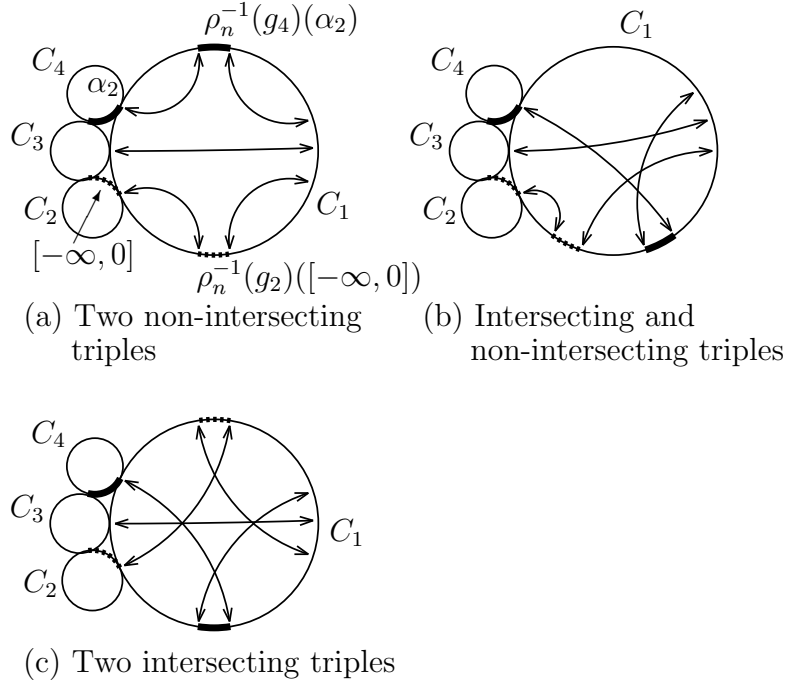


FIGURE 6

Thus, the ratios of the radius of C_5 and C_6 to that of C_4 , $\frac{rad(C_5)}{rad(C_4)}$ and $\frac{rad(C_6)}{rad(C_4)}$ approach zero and the cross ratios of the two edges v_1v_4 and v_3v_4 both approach ∞ . Repeating the argument with the roles of C_2 and C_4 reversed, we see that either $|tr(\varphi_n)| \rightarrow \infty$ or the cross ratios of v_1v_2 and v_2v_3 both approach ∞ . In other words, either $|tr(\varphi_n)| \rightarrow \infty$ or the cross ratios of all neighbor edges approach ∞ . By induction, using the edge e_4 instead of e_3 and repeating the above argument, we see that either some element $g \in F$ has holonomy with diverging trace, or the cross ratios of all the edges of τ diverge to ∞ . However, in the latter case, the $(2, 2)$ term d_m of the matrix corresponding to the word W_m defined in §3 has a dominating term $x_1x_2 \cdots x_m$ and diverges to ∞ . This contradicts the fact that $d_m = -1$ for all points of the sequence by condition (1) in §3. This completes the proof. \square

Lemma 5.1 implies

Lemma 5.2. *Suppose that τ has only one vertex. The composition $\pi \circ f : \mathcal{C} \rightarrow \mathcal{T}$ of the forgetting map with the collapsing map $\pi : \mathcal{P} \rightarrow \mathcal{T}$ in the Thurston construction is proper. In particular, the restriction of π to \mathcal{I} is proper.*

Proof. Suppose that $\{\mathbf{c}_n\}$ is a sequence in \mathcal{C} which escapes away from the compact sets. By passing to a subsequence, we may assume by lemma 5.1 that there is an element $g \in \pi_1(\Sigma)$ with holonomy $\varphi_n = \rho_n(g)$ for which $|\text{tr}(\varphi_n)| \rightarrow \infty$. g corresponds to a closed curve γ on Σ and the length $l_n(\gamma)$ of the geodesic representative of γ on the collapsed surface $H_n = \pi(S_n)$ satisfies the inequality

$$l_n(\gamma) \geq d_{\mathbb{H}^3}(z_n, \varphi_n(z_n))$$

where z_n is any point on the axis of φ_n in \mathbb{H}^3 and $d_{\mathbb{H}^3}(z_n, \varphi_n(z_n))$ is the hyperbolic distance between z_n and $\varphi_n(z_n)$. Since $|\text{tr}(\varphi_n)| \rightarrow \infty$, $d_{\mathbb{H}^3}(z_n, \varphi_n(z_n)) \rightarrow \infty$ and hence $l_n(\gamma) \rightarrow \infty$. It follows that $\{H_n\}$ escapes away from the compact sets in the Teichmüller space \mathcal{T} . \square

6. PROOF OF THEOREM

We recall the following result from [11], the notation has been modified slightly to fit into our framework.

Theorem 6.1 ([11], theorem 3.4). *Let $S = (H, \lambda)$ be a surface with projective structure where $H = \pi(S) \in \mathcal{T}$ and $\lambda = \beta(S) \in \mathcal{ML}$ and let X be the underlying Riemann surface. Let $h : X \rightarrow H$ denote the harmonic map with respect to the hyperbolic metric on H and $\mathcal{E}(h)$ its energy. Then*

$$l_H(\lambda) \leq \frac{l_H(\lambda)^2}{E_X(\lambda)} \leq 2\mathcal{E}(h) \leq l_H(\lambda) + 8\pi(g-1), \quad (5)$$

where $l_H(\lambda)$ is the hyperbolic length of λ on H , and $E_X(\lambda)$ is the extremal length of λ on X .

We are now ready to prove the theorem.

Proof of Theorem 1.1. Assuming that τ has one vertex, we follow the argument of Tanigawa in [11]. Let $\{\mathbf{c}_n\}$ be a sequence of points in \mathcal{C} which escapes away from the compact sets, $f(\mathbf{c}_n) = S_n = (H_n, \lambda_n) \in \mathcal{T} \times \mathcal{ML}$ and $X_n = u(S_n)$ the corresponding underlying Riemann surface. By lemma 5.2, we may assume that $\{H_n\}$ escapes away from the compact sets in \mathcal{T} . Furthermore, by lemma 4.1, since $\{\lambda_n\}$ lies in a compact subset of \mathcal{ML} , we may assume by taking a subsequence if necessary that $\lambda_n \rightarrow \lambda$ for some fixed measured lamination λ . Let h_n be the harmonic map from X_n to H_n . We consider the two cases

- (i) $\sup_n l_{H_n}(\lambda_n) < \infty$, or
- (ii) $\lim_{n \rightarrow \infty} l_{H_n}(\lambda_n) = \infty$.

First assume that case (i) holds. If $\{X_n\}$ stays in a compact subset of \mathcal{T} , then since $\{H_n\}$ escapes away from the compact sets, by a result of

M. Wolf [13], $\mathcal{E}(h_n) \rightarrow \infty$. This contradicts (i) and the right inequality of (5). Hence $\{X_n\}$ escapes away from the compact sets.

Next suppose that case (ii) holds. Then by (5)

$$\lim_{n \rightarrow \infty} E_{X_n}(\lambda_n) = \lim_{n \rightarrow \infty} E_{X_n}(\lambda) = \lim_{n \rightarrow \infty} (l_{H_n}(\lambda_n) + O(1)) = \infty,$$

and so $\{X_n\}$ escapes away from the compact sets as well and we are done. \square

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