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Circle packings on surfaces with projective structures: A survey

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Abstract

This paper surveys our on-going study of the moduli space of pairs of a surface with a complex projective structure, on which the circle makes sense, and a circle packing on it whose combinatorics is fixed. A conjectural picture, the results obtained so far and a list of problems for further study are discussed.

1. Introduction

The study of circle packings dates back to antiquity, but has seen a surge of activity in the last thirty years, especially after Thurston's [Thu78] re-interpretation and generalization of the circle packing theorem on the sphere by Koebe [Koe36] and Andreev [And70] to surfaces of higher genus. Classically, this involves the study of circle patterns as defined on a surface with a Riemannian metric. Circle packings have found applications to various fields including complex analysis, hyperbolic geometry and even probability theory. On the other hand, the study of surfaces with complex projective structures, called *projective Riemann surfaces* in this paper, is relatively modern (we avoid the terminology projective surface as it has a different meaning in algebraic geometry, the terminology $\mathbb{CP}^1$-surface is also used in the literature). It has also seen much activity in the last couple of decades due to its connections with hyperbolic geometry, Kleinian groups and Teichmüller theory.

We are interested in the interplay between these two fields, specifically, circle packings on projective Riemann surfaces. The first observation is that circles/disks are fundamental geometric objects in 1-dimensional complex projective geometry. This is despite the fact that $\mathrm{PGL}_2(\mathbb{C})\,(=\mathrm{PSL}_2(\mathbb{C}))$ does not preserve a spherical metric (in fact, any metric) on the Riemann sphere $\widehat{\mathbb{C}}$. Thus "circles" on a projective Riemann surface are not metric circles in the usual sense, rather, they are homotopically trivial simple closed curves which develop onto (round) circles in $\widehat{\mathbb{C}}$ via the developing map. This is an important distinction as methods which rely on parameters such as the radii of the circles do not come into play here. Circles on a projective Riemann surface are well-defined, since $\mathrm{PGL}_2(\mathbb{C})$ acts as Möbius transformations of $\hat{\mathbb{C}}$ and Möbius transformations preserve circles.

Our main focus will be on the moduli space of all pairs (S,P) consisting of a projective Riemann surface S, and a circle packing P on S with a fixed combinatorial pattern. Using the Koebe-Andreev-Thurston theorem as a starting point and prototype, and setting up a conjectural picture which we will discuss in §5 based on the second author's result [Miz00], we studied the intrinsic and extrinsic properties of such moduli spaces in [KMT03, KMT]. This paper surveys these on-going studies by describing a foundational basis, the results obtained so far, and a list of problems for further study.

2. Projective Structures

Throughout this paper, we use Σ_g to denote a closed, orientable topological surface of genus g without any auxiliary structure. A projective structure on Σ_g is a geometric structure modeled on the pair of the Riemann sphere $\widehat{\mathbb{C}}$ and the projective linear group $\mathrm{PGL}_2(\mathbb{C})$ acting on $\widehat{\mathbb{C}}$ by orientation preserving projective transformations, that is, the Möbius transformations. Since Möbius transformations are in particular holomorphic and one-to-one, a projective structure automatically induces an underlying complex structure. However, requiring the transition maps to be Möbius transformations is far more rigid than merely requiring that they be holomorphic one-to-one maps, so different projective structures can have the same underlying complex structure.

Let S be a surface with a projective structure, called a projective Riemann surface, homeomorphic to Σ_g. The notation S is thus to denote not just the topological surface Σ_g, but one equipped with a projective structure. We always attach to S a reference homeomorphism $h : \Sigma_g \to S$ for marking. Then two projective Riemann surfaces, say (S_1, h_1) and (S_2, h_2), are considered to be marked projectively equivalent if there exists a projective isomorphism $\varphi : S_1 \to S_2$ such that $\varphi \circ h_1$ is homotopic to h_2.

Associated to S are the developing map,

$$\mathrm{dev} : \widetilde{S} \longrightarrow \widehat{\mathbb{C}},$$

defined up to composition with projective transformations, where $\widetilde{S}$ is the universal cover of S, and the holonomy representation,

$$\rho : \pi_1(S) \longrightarrow \mathrm{PGL}_2(\mathbb{C}),$$

defined up to conjugation by projective transformations. The developing map is equivariant with respect to the holonomy representation, that is,

$$\mathrm{dev}(x.g) = (\mathrm{dev}(x)).\rho(g)$$

for all $g \in \pi_1(S)$, $x \in \tilde{S}$, where $\pi_1(S)$ acts as deck transformations on $\tilde{S}$. We note that both the developing map and the holonomy representation can be extremely complicated. In general, the developing map is not injective to its image and the holonomy representation may not be discrete or faithful. One of the main questions arising in this theory is the question of which representations can occur as the holonomy of a projective structure. This was settled by Gallo, Kapovich and Marden in [GKM00], where they showed that any representation whose image is non-elementary and which lifts to $SL_2(\mathbb{C})$ occurs as the holonomy representation of some projective structure.

There are many natural projective structures, for example, the canonical complex structure on the Riemann sphere. This projective structure on the sphere is unique, since the conformal automorphisms of the Riemann sphere with the underlying complex structure are precisely $PGL_2(\mathbb{C})$. Similarly, a Euclidean structure on the torus also naturally defines a projective structure, since Euclidean isometries form a subgroup of $PGL_2(\mathbb{C})$, and the Euclidean plane is a subset of the Riemann sphere. Likewise, a hyperbolic structure on Σ_g defines a projective structure, for the analogous reason that the isometries of $\mathbb{H}^2$ can be identified with a subgroup of $PGL_2(\mathbb{C})$. More interesting examples arise from, say, taking the quotient of one of the components of the domain of discontinuity of a quasi-Fuchsian group by the group, or by performing grafting on a hyperbolic structure, see for example [Gol87].

3. Circle Packings

A projective structure on a surface is not a metric structure, but the circle still makes sense since a projective (Möbius) transformation maps a circle on $\widehat{\mathbb{C}}$ to a circle on $\widehat{\mathbb{C}}$. Hence we define a circle on a projective Riemann surface S to be a homotopically trivial curve on S such that its lifts on $\widetilde{S}$ are mapped to circles in $\widehat{\mathbb{C}}$ by the developing map.

Let S be a projective Riemann surface. A circle packing P on S is a collection of circles in S such that each circle bounds a disk and the interior of the disks are all disjoint. The circle packings we will consider hereafter are also assumed to have the additional property:

$$\text{Complementary regions are all triangular.} \tag{3.1}$$

Note that the circles are allowed to have self-tangency points.

To each circle packing, we associate a graph τ on S called the nerve of P. It is obtained by assigning a vertex to each circle and an edge between two (not necessarily distinct) vertices for each tangency point. It is easy to see that τ is covered by an honest triangulation $\widetilde{\tau}$ in the universal cover $\widetilde{S}$.

Conversely, suppose we are given a topological graph τ on Σ_g such that τ is covered by an honest triangulation in the universal cover. We then would like to find a complete description of the set of all pairs (S,P), where S is a projective Riemann surface equipped with a reference homeomorphism $h : \Sigma_g \to S$, and P is a circle packing on S such that its nerve is isotopic to $h(\tau)$. Here two packings P_1 and P_2 on the same surface S are equivalent if there is a projective automorphism of S isotopic to the identity which takes P_1 to P_2, and we are interested in the equivalence classes of packings. Henceforth, for clarity of exposition, we shall simply say that P has nerve τ, or nerve $h(\tau)$, as the case may be, and suppress mentioning isotopy.

We can ask the same question for Riemannian surfaces of constant curvature, that is, to give a complete description of pairs (S', P') where S' is a surface with a constant curvature Riemannian metric and P' is a circle packing on S', consisting of circles with respect to the Riemannian metric, and whose nerve is isotopic to τ. Indeed, this is the context addressed in most papers on circle packings. Here two packings P'_1 and P'_2 on the same Riemannian surface S' are equivalent if there exists a conformal automorphism of the surface (with respect to the Riemannian metric) isotopic to the identity which takes P'_1 to P'_2, and we are interested in the equivalence classes of circle packings. Then the Koebe-Andreev-Thurston theorem [Koe36, And70, Thu78] says that given τ, there is a unique such pair (S', P') up to scalar multiple and isotopy of the metric. In the case where $g = 0$, we assume the metric is scaled to have constant curvature 1 and where $g \geq 2$ we assume that the metric has been scaled to have constant curvature -1. More precisely, when $g = 0$, there is a circle packing P on the unit sphere with nerve τ, unique up to conformal automorphisms of the sphere (that is, up to the action of $\mathrm{PGL}_2(\mathbb{C})$ on the Riemann sphere). In the case where $g = 1$, there is a unique Euclidean torus up to scaling with a circle packing with nerve τ and the packing is rigid up to translation. And when $g \geq 2$, there is a unique hyperbolic surface with a circle packing with nerve τ and the packing is rigid. Observe that the solution (S', P') also provides a solution (S, P) to our original question in the category of projective Riemann surfaces. We shall call this solution the KAT pair associated to τ. The main interest of our study is to see how much the KAT pair for τ can be deformed with a deformation of the projective structure.

Our first clue that there was a rich deformation theory arose from results of computer experiments. Figures 1 and 2 show the developing images of two projective circle packings on a genus 2 surface which have the same nerve τ (up to isotopy). This is an example where τ has exactly one vertex. Figure 1 represents the KAT solution for τ, figure 2 a small deformation. In both cases the developing maps are injective, but there are also many examples where the developing map is not injective.

4. Cross Ratios

Projective geometry is not a metric geometry and metric concepts such as radii do not make sense in general. Hence we need to use some other invariants which would help to quantify and parameterize the space of pairs (S, P) and allow us to analyze the space more systematically. We introduce a projective invariant of a circle packing based on the cross ratio. It can also be found in the works by He and Schramm [HS98] and separately by Schramm [Sch97] as well in a different context.

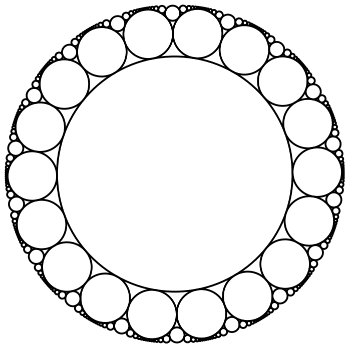

Figure 1: Hyperbolic example

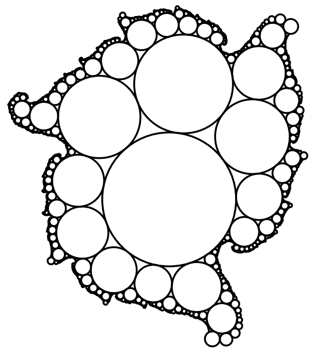

Figure 2: Deformed example

Suppose that (S,P) is a pair of a projective Riemann surface S and a circle packing P on S with the property (3.1). The invariant of (S,P) which we will define is a map

$$\mathbf{x} : E_\tau \longrightarrow \mathbb{R},$$

where E_τ is the set of edges of τ. To each edge e of τ, we choose a lift $\widetilde{e}$ in $\widetilde{\tau}$ and associate a configuration of four circles on $\widehat{\mathbb{C}}$ in the developed image about $\mathrm{dev}(\widetilde{e})$, see Figure 3. Recall that the cross ratio of four distinct ordered points in $\hat{\mathbb{C}}$ is given in [Ahl53] by

$$(z_1,z_2,z_3,z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}.$$

It is the value of the image of z_1 under the projective transformation which takes z_2, z_3 and z_4 to $1, 0$ and ∞ respectively. The value assigned to the edge e will be the imaginary part of the cross ratio of the four contact points $(p_{14}, p_{23}, p_{12}, p_{13})$ of the configuration chosen as in Figure 3 with orientation convention. Note that the cross ratio of these four points is always purely imaginary with positive imaginary part since the projective transformation taking the ordered triple (p_{23}, p_{12}, p_{13}) to $(1,0,\infty)$ maps C_1 to the imaginary axis, and hence takes p_{14} to a point on the positive imaginary axis due to the nature of the configuration. Since the cross ratio is a projective invariant, the value does not depend on the choice of lift $\widetilde{e}$ and the developing map. Collecting the values for each edge, we obtain the map $\mathbf{x}$ of E_τ, which we call a cross ratio parameter. The cross ratio of the edge e determines the position of the circle C_4 in Figure 3 once the positions of C_1, C_2 and C_3 are fixed, and if the cross ratio of e approaches ∞, then C_4 approaches p_{13}.

Of course, not all real valued maps of E_τ can be the cross ratio parameter for some circle packing. There must be some conditions which ensure that for each circle in $\tilde{S}$, the developing image of the surrounding circles closes up neatly. The key to obtaining these conditions in a concise and useful form is to consider a normalized picture of

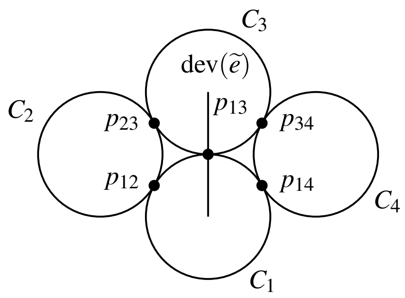

Figure 3: Four circle configuration

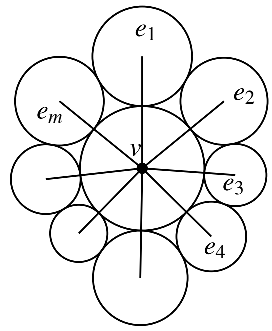

Figure 4: A surrounded circle

a circle with its surrounding circles. The normalization we chose maps the central circle to the real line and one of the adjoining interstices to the standard interstice with vertices at ∞, 0 and $\sqrt{-1}$. This led us to introduce an associated matrix $A \in \mathrm{SL}_2(\mathbb{R})$ to each edge $e \in E_\tau$. If the value of a cross ratio parameter at e is x, A is defined to be $\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. The relationship between the associated matrix and the configuration of four circles corresponding to an edge with cross ratio x can then be seen by first normalizing the configuration in Fig 3 by sending the triple (p_{12}, p_{13}, p_{23}) to $(\infty, 0, \sqrt{-1})$, so that we are considering C_1 as the central circle and taking the normalized picture. In the normalized configuration, C_1 and C_2 are horizontal lines through 0 and $\sqrt{-1}$ respectively, C_3 is a circle of radius $1/2$ with center at $\sqrt{-1}/2$ and C_4 is a circle tangent to C_1 at the point $1/x$, and is also tangent to C_3. Then a simple computation shows that the associated matrix A represents a transformation which sends the left triangular interstice of this configuration to the right triangular interstice.

Let v be a vertex of τ with valence m. We read off the edges $e_1, \cdots, e_m$ incident to v in a clockwise direction to obtain a sequence of assigned values $x_1, \ldots, x_m$ of cross ratio parameters. Let

$$W_j = A_1 A_2 \cdots A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad j = 1, \ldots, m,$$

where A_i is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & x_i \end{pmatrix}$ associated to e_i. Then, by a careful study of the normalized picture, and the composition of moves that shifts the standard interstice to the interstices on its right step by step, until it finally returns to itself, it was verified in [KMT03] that for each vertex v of τ, we have

$$W_v = A_1 A_2 \cdots A_m = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.1}$$

and

$$\begin{cases} a_j, c_j < 0,\ b_j, d_j > 0 \ \text{ for } \ 1 \le j \le m-1 \\ \text{except for } \ a_1 = d_{m-1} = 0. \end{cases} \tag{4.2}$$

The first condition comes from the fact that the chain of circles surrounding the circle corresponding to v closes up. The second condition is a no overwinding condition, and it eliminates the case where the chain surrounds the central circle more than once. Notice here that the associated matrices are in $\mathrm{SL}_2(\mathbb{R})$ and not in $\mathrm{PSL}_2(\mathbb{R})$, so that the inequalities of (4.2) do make sense.

On the other hand, given a real valued map $\mathbf{x}$ of E_τ satisfying (4.1) and (4.2) for each vertex of τ, it is relatively routine to construct a pair (S,P) of a projective Riemann surface S and a circle packing P on S so that its cross ratio parameter is $\mathbf{x}$ (see [KMT03] for details). We thus set

$$\mathscr{C}_\tau = \{\mathbf{x} : E_\tau \to \mathbb{R} \,|\, \mathbf{x} \ \text{satisfies (4.1) and (4.2) for each vertex}\},$$

and call it the cross ratio parameter space.

Since the condition (4.1) gives a set of polynomial equations for the x_i's and (4.2) are polynomial inequalities in the x_i's, $\mathscr{C}_\tau$ is a semi-algebraic set by definition, and we define the topology on $\mathscr{C}_\tau$ to be the one induced by the tautological inclusion $\iota : \mathscr{C}_\tau \to \mathbb{R}^{E_\tau}$. It turns out that this naive construction gives us a correct parameterization of the moduli space of pairs (S,P) where S is a projective Riemann surface and P is a circle packing on S with nerve τ.

Lemma 4.1. *We have the following :*

(i) **(Lemma 2.17 in [KMT03])** $\mathscr{C}_\tau$ *corresponds bijectively to the moduli space of all pairs* (S,P) *where S is a projective Riemann surface and P is a circle packing on S combinatorially with nerve* τ*, up to marked projective equivalence.*

(ii) **(Lemma 3.2 in [KMT])** *The tautological inclusion* $\iota : \mathscr{C}_\tau \to \mathbb{R}^{E_\tau}$ *is proper.*

The above result states that we can identify $\mathscr{C}_\tau$ with the moduli space of all pairs (S,P) with nerve τ, which we do from now on. The study of the moduli space then reduces to the study of the semi-algebraic set $\mathscr{C}_\tau$.

5. Conjectures

Since the KAT pair represents a point in $\mathscr{C}_\tau$, the moduli space is certainly nonempty. However, this is the only fact we know up to this stage, and we are far from knowing what $\mathscr{C}_\tau$ looks like. To understand $\mathscr{C}_\tau$ better, we relate it with some other spaces and formulate a conjectural picture.

Let $\mathscr{P}_g$ be the space of all projective Riemann surfaces homeomorphic to Σ_g up to marked projective equivalence. In other words, it is the space of all marked projective structures on Σ_g. To each pair (S,P) in $\mathscr{C}_\tau$, assign only its first component and we obtain the forgetting map

$$f : \mathscr{C}_\tau \longrightarrow \mathscr{P}_g.$$

Thus the image $f(\mathscr{C}_\tau)$ consists of all projective Riemann surfaces which admit a circle packing with nerve τ and it is not difficult to see that the injectivity of f is equivalent to the rigidity of the circle packings with nerve τ on such projective Riemann surfaces.

Let $\mathscr{T}_g$ be the space of all complex structures on Σ_g up to marked conformal equivalence, which is commonly called the Teichmüller space. To each projective Riemann surface, assign its underlying complex structure and we obtain the projection map

$$p : \mathscr{P}_g \longrightarrow \mathscr{T}_g.$$

Teichmüller space is known to be homeomorphic to the Euclidean space of dimension 2 or $6g-6$ according to whether $g = 1$ or $g \geq 2$. The projection map p is a vector bundle projection where the fiber over each conformal class consists of its holomorphic quadratic differentials, and each fiber is a vector space of the same dimension as $\mathscr{T}_g$. In particular, $\mathscr{P}_g$ is homeomorphic to Euclidean space of dimension 4 or $12g-12$ according to whether $g = 1$ or $g \geq 2$.

To each conformal class of a Riemann surface homeomorphic to Σ_g, the Koebe-Riemann Uniformization Theorem states that there is a unique metric of constant curvature $1, 0$ or -1 on the surface, up to scaling if $g = 1$, in the conformal class. Hence, we obtain a natural section

$$s : \mathscr{T}_g \longrightarrow \mathscr{P}_g$$

to the projection map $p : \mathscr{P}_g \to \mathscr{T}_g$. This section is none other than the space of marked Euclidean structures on the torus up to scaling in the case where $g = 1$, and the space of marked hyperbolic structures on Σ_g in the case $g \geq 2$, which, we recall from §2, are projective structures by definition.

The Koebe-Andreev-Thurston theorem implies that $f(\mathscr{C}_\tau)$ intersects $s(\mathscr{T}_g)$ only at $f(\{\text{KAT}\})$ and furthermore, the rigidity of the circle packing on $f(\{\text{KAT}\})$ means that the inverse image of this point under f consists of exactly one point. It is natural to conjecture that the rigidity of the circle packings holds for all projective Riemann surfaces in $f(\mathscr{C}_\tau)$, that is, each projective Riemann surface S admits at most one circle packing with nerve τ up to projective automorphisms of S isotopic to the identity. This can be stated as follows:

Conjecture 1. *The forgetting map $f : \mathscr{C}_\tau \to \mathscr{P}_g$ is injective.*

This does not however give a possible description of the space $\mathscr{C}_\tau$. To get a more detailed understanding of $\mathscr{C}_\tau$ and its image under f, we formulate a stronger conjecture, which certainly implies the first one, as follows:

Conjecture 2. *The composition $p \circ f : \mathscr{C}_\tau \to \mathscr{T}_g$ is a homeomorphism.*

The motivation for this rather strong conjecture goes back to the second author's result in [Miz00], where it was shown to be true when $g = 1$ and τ has only one vertex. Here are some implications of the conjecture, which we believe to be true, and have verified in certain special cases. They serve as further motivation for the conjecture:

(i) The moduli space $\mathscr{C}_\tau$ is homeomorphic to Euclidean space of dimension 2 or $6g-6$ according to whether $g = 1$ or $g \geq 2$.

(ii) $f(\mathscr{C}_\tau)$ and $s(\mathscr{T}_g)$ are middle dimensional proper submanifolds in $\mathscr{P}_g$ which intersect only at $f(\{\mathrm{KAT}\})$. Probably the intersection can be shown to be transverse with a little more argument.

(iii) The image $f(\mathscr{C}_\tau)$ of the forgetting map defines a new natural section or a slice to $p : \mathscr{P}_g \to \mathscr{T}_g$. In words, this would mean that for each conformal class of a Riemann surface, there exists a unique projective structure within the conformal class which admits a circle packing with nerve τ.

6. Local Results

We have shown that part of the conjecture is true, at least topologically in a neighborhood of the KAT pair.

Theorem 6.1 (Theorem 1 in [KMT03])**.** *There is a neighborhood U of the* KAT *pair in $\mathscr{C}_\tau$ such that*

(i) *U is homeomorphic to Euclidean space of dimension* 2 *or* $6g-6$ *according to whether $g = 1$ or $g \geq 2$,*

(ii) *the restriction of f to U is injective.*

This result was proved by comparing the deformation of a hyperbolic 3-manifold constructed from the KAT pair and the deformation of a projective Riemann surface admitting a circle packing with nerve τ. Hyperbolic Dehn filling theory and quasi-Fuchsian deformation theory were used for the cases $g = 1$ and $g \geq 2$ respectively.

Since the neighborhood of U is defined constructively, we roughly know how large it is. When $g = 1$, U is chosen so that the image of U under f is identified with hyperbolic Dehn surgery space of a corresponding cusped 3-manifold, which can be embedded as a 2-dimensional subspace in $\mathscr{P}_1$. Since hyperbolic Dehn surgery space omits only a finite number of classical Dehn surgery coefficients, U in this case would be fairly large. When $g \geq 2$, U is chosen to be the preimage of the space of all quasi-Fuchsian deformations of a corresponding hyperbolic 3-manifold by f. Since this space can be identified with the connected component of discrete representations containing the one coming from the KAT pair, U in this case is also fairly large.

7. Global Results

The work to prove the global results stated in Conjectures 1 and 2 is still in progress. We discuss in this section some of the partial results we have obtained thus far in [KMT03] and [KMT]. To begin, it is useful to consider Thurston's parameterization of $\mathscr{P}_g$ which we will describe shortly, since it is more geometric than the projection map $p : \mathscr{P}_g \to \mathscr{T}_g$. We first describe two more spaces closely related with $\mathscr{P}_g$.

The first is the space of non-elementary representations of $\pi_1(\Sigma_g)$ in $\mathrm{PGL}_2(\mathbb{C})(= \mathrm{PSL}_2(\mathbb{C}))$ up to conjugation, which has a natural structure of a $(6g-6)$-dimensional complex analytic manifold [Gun67]. Since the holonomy representation of a projective structure is not only non-elementary but lifts to $\mathrm{SL}_2(\mathbb{C})$, we will focus on the open subset $\mathscr{X}_g$ consisting of representations which lift to $\mathrm{SL}_2(\mathbb{C})$ up to conjugation. Then, assigning its holonomy representation to each projective Riemann surface, we obtain the map

$$hol : \mathscr{P}_g \longrightarrow \mathscr{X}_g.$$

hol is known to be a local homeomorphism by Hejhal [Hej75].

The second is the space of isotopy classes of measured laminations on Σ_g $(g \geq 2)$, which we denote by $\mathscr{ML}_g$. A measured lamination is defined to be a closed subset on Σ_g locally homeomorphic to a product of a totally disconnected subset of the interval with an interval, together with a transverse measure. A simple closed curve on Σ_g with the counting measure for transverse arcs is an elementary, but important and fundamental example of a measured lamination. In fact, the set of weighted simple closed curves is dense in $\mathscr{ML}_g$. See [Thu78, Thu88] for details.

Although a measured lamination is a topological concept, once we put a hyperbolic metric on Σ_g, its support is canonically realized as a disjoint union of simple geodesics which forms a closed subset on the surface. Such a lamination is called a geodesic lamination with transverse measure.

We now describe Thurston's parameterization of $\mathscr{P}_g$. Thurston has shown that any projective Riemann surface corresponds uniquely to a hyperbolic surface pleated along a geodesic lamination with a fixed bending measure. Following [KT92], we briefly review his parameterization. Start with a projective structure on S which is not a hyperbolic structure. Consider the set of maximal disks in the universal cover $\widetilde{S}$. Each maximal disk is naturally endowed with the hyperbolic metric, the boundary of each disk intersects the ideal boundary of $\widetilde{S}$ in two or more points and we can take the convex hull of these ideal boundary points. It can be shown that this gives a stratification of $\widetilde{S}$ by ideal polygons, and ideal bigons foliated by "parallel lines" joining the two ideal vertices of the bigons. The polygonal parts support a canonical hyperbolic metric. Collapsing each bigon foliated by parallel lines in $\widetilde{S}$ to a line and taking the quotient of the result by the action of the fundamental group, we obtain a hyperbolic surface H. This defines a hyperbolization map

$$\pi : \mathscr{P}_g \to s(\mathscr{T}_g).$$

Also the stratification defines a geodesic lamination λ on H by taking the union of collapsed lines. Moreover, using the convex hull of the ideal points of the maximal disk not in the disk but in the 3-dimensional hyperbolic space, we can assign a transverse bending measure supported on λ. This defines a pleating map

$$\beta : \mathscr{P}_g \to \mathscr{ML}_g.$$

The pair of these maps (π, β) becomes a homeomorphism of $\mathscr{P}_g$ onto $s(\mathscr{T}_g) \times \mathscr{ML}_g$.

Figure 5 shows the related spaces and the maps between them as discussed above.

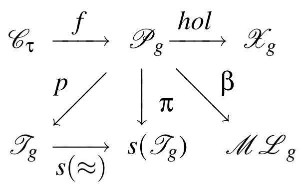

Figure 5: Related Spaces

Now by using Thurston's parameterization of $\mathscr{P}_g$, we can analyze $f : \mathscr{C}_\tau \to \mathscr{P}_g$ by looking at $\pi \circ f$ and $\beta \circ f$ separately. We have the following result:

Lemma 7.1 (Lemma 4.1 in [KMT])**.** *If $g \geq 2$, then the composition $\beta \circ f : \mathscr{C}_\tau \to \mathscr{ML}_g$ has bounded image.*

This is proved by observing how the developed image of a projective Riemann surface admitting a circle packing with nerve τ is controlled by the combinatorial data of τ. The key is that there is a relation between the circle packing on $dev(\tilde{S})$ and the maximal disks in $dev(\tilde{S})$ which can be exploited to control the image $\beta \circ f(\mathscr{C}_\tau)$.

Apart from this general result, the best global results towards the conjecture we have obtained so far are for the case when τ has only one vertex. This arose from our attempt to understand the cross ratio parameter space $\mathscr{C}_\tau$ concretely in the simplest settings. Note that in this case, $\mathscr{C}_\tau$ is defined by just one matrix equation and set of inequalities corresponding to (4.1) and (4.2) respectively.

Theorem 7.2. *If τ has one vertex and $g \geq 2$, then*

(i) **(Theorem 2 and Lemma 5.1 in [KMT03])** $\mathscr{C}_\tau$ *is homeomorphic to* $\mathbb{R}^{6g-6}$ *and* $hol \circ f : \mathscr{C}_\tau \to \mathscr{X}_g$ *is injective. In particular,* $f : \mathscr{C}_\tau \to \mathscr{P}_g$ *is injective.*

(ii) **(Theorem 1.1 in [KMT])** $p \circ f : \mathscr{C}_\tau \to \mathscr{T}_g$ *is proper.*

Theorem 7.2 comes fairly close to proving Conjecture 2 for the one circle packing case. What is missing is a proof that p restricted to $f(\mathscr{C}_\tau)$ is locally injective. An argument similar to the one worked out by Scannell and Wolf in [SW02] is expected to complete this case.

The arguments used to prove Theorem 7.2 depend technically on the simplifying assumption that τ has only one vertex. At the moment, we do not know how to generalize these arguments to prove the conjecture in general. In fact, the proof of Theorem 7.2 (i) given in [KMT03] involves a careful study of the cross ratio parameter space, showing that one can always choose a set of $6g-6$ free parameters lying in a convex subset of $\mathbb{R}^{6g-6}$ which completely parameterizes $\mathscr{C}_\tau$. This requires a very good understanding of the equations (4.1) and inequalities (4.2). Further analysis then shows that in fact $hol \circ f$ is injective, from which we conclude that $f : \mathscr{C}_\tau \to \mathscr{P}_g$ is injective.

The proof of Theorem 7.2 (ii) uses Theorem 7.2 (i), but otherwise does not rely on the assumption that τ has only one vertex. The main ingredient is the result of Tanigawa in [Tan97] which relates quantities associated to $p \circ f$, $\pi \circ f$, and $\beta \circ f$. We give a brief sketch of the proof. First note that the fact that $hol \circ f$ is injective, stated in Theorem 7.2 (i), immediately implies that $\pi \circ f$ is proper. Now choose a sequence x_n in $\mathscr{C}_\tau$ which escapes every compact subset. Since $\pi \circ f$ is proper, the hyperbolic surfaces $H_n = \pi \circ f(x_n)$ escape every compact subset of $\mathscr{T}_g$. Let $h_n : X_n \to H_n$ be the harmonic map, where X_n is the Riemann surface $p \circ f(x_n)$. Then Tanigawa's inequalities [Tan97] imply that either the extremal lengths $E_{\lambda_n}(X_n)$ of $\lambda_n = \beta \circ f(x_n)$ in X_n diverge, or that the energies of the maps h_n remain bounded. In the first case, since by Lemma 7.1 λ_n is bounded in $\mathscr{ML}_g$, it must be that X_n escapes every compact subset. In the second case, the result of Wolf in [Wol89] implies that X_n is again unbounded. This concludes the proof that $\pi \circ f$, and hence $p \circ f$ are proper.

8. Cone Projective Structures

This section is just to give a small remark not mentioned in [KMT03, KMT]. It is natural to study the extension of circle packings on projective Riemann surfaces to cone projective Riemann surfaces, where the cone points are in the vertex set of τ, as was done in the metric structure case. This would include circle packings on orbifold surfaces and branched surfaces, and can also be used to connect a circle packing on a compact surface and a horocycle packing on a cusped surface continuously.

The analysis using the cross ratio parameter can be applied to the study of circle packings on projective Riemann surfaces with cone singularities, with little modification. In the case of cone structures, the equation (4.1) is replaced by an equation involving the trace, arising from the cone angle condition assuming the cone points are "centers" of circles. However, it can be much more complicated if we allow cone angles $> 2\pi$. To demonstrate the complications, we describe here the moduli space of circle packings by one circle on the torus with a cone point where the cone point is the center of the circle.

Let θ (≥ 0) be a cone angle and τ the nerve of a circle packing. We here regard a cone point with $\theta = 0$ as a cusp. Let us denote the cross ratios corresponding to the three edges of τ by x, y, z and their associated matrices by X, Y, Z. Then the condition corresponding to (4.1) in this case is replaced by the condition that $(XYZ)^2$ is conjugate to a θ-rotation when $\theta > 0$ and a translation when $\theta = 0$. This implies the equation,

$$|\mathrm{tr}(XYZ)| = |2\cos(\theta/4)|,$$

by which we can compute the cross ratio parameter space $\mathscr{C}_{\tau,\theta}$ concretely as follows.

$$\mathscr{C}_{\tau,\theta} = \begin{cases} \{xyz - x - y - z = 2,\ xy > 1,\ x > 0\} & \text{if } \theta = 0. \\ \{xyz - x - y - z = 2\cos\frac{\theta}{4},\ xy > 1,\ x > 0\} & \text{if } \theta \in (0, 4\pi). \\ \{(1,1,1)\} & \text{if } \theta = 4\pi. \\ \{xyz - x - y - z = 2\cos\frac{\theta}{4},\ xy < 1\} & \text{if } \theta \in (4\pi, 8\pi). \\ \{(-1,-1,-1)\} & \text{if } \theta = 8\pi. \\ \{xyz - x - y - z = 2\cos\frac{\theta}{4},\ xy > 1,\ x < 0\} & \text{if } \theta \in (8\pi, 12\pi). \end{cases}$$

When $\theta = 0$, 4π and 8π, the trace condition is not sufficient to determine the conjugacy class of XYZ, and we need a little argument to obtain the above representations.

The equation $xyz - x - y - z = t$ has the unique solution z for given x, y, t unless $xy = 1$, and $\mathscr{C}_{\tau,\theta}$ is homeomorphic to $\mathbb{R}^2$ except for $\theta = 4\pi, 8\pi$. Note that it degenerates to a point in these two exceptional cases.

9. Problems

Here we list up some open problems and directions for further study in the subject.

(i) Can we use a variational method to study the conjecture, similar to the methods used by Colin de Verdiere in [Col91] ?

(ii) If the conjecture were true, the resulting new section in $\mathscr{P}_g$ should find some applications in the study of Kleinian group theory. What are they ?

(iii) In the one circle packing case, the moduli space $\mathscr{C}_\tau$ can be identified with a very nice convex subset of $\mathbb{R}^{6g-6}$. This is as opposed to many other embeddings of $\mathscr{T}_g$ which have fractal type boundary. Do $\mathscr{C}_\tau$ or $f(\mathscr{C}_\tau)$ have nice compactifications with interesting geometric interpretations ?

(iv) Brooks [Bro86] has shown that the set of hyperbolic surfaces admitting a circle packing with the property (3.1) is dense in Teichmüller space $\mathscr{T}_g$. A natural question is whether projective Riemann surfaces admitting a circle packing with the property (3.1) are dense in $\mathscr{P}_g$?

(v) Study the intersection of $f(\mathscr{C}_\tau)$ with the space of quasi-Fuchsian structures for fixed g and τ. For example, we may ask whether this intersection is connected, and what is the boundary of this space like ?

(vi) Study the geometry of $\mathscr{P}_g$ from $f(\mathscr{C}_\tau)$ for various τ.

(vii) The cross ratios are naturally positive real numbers, but it is possible to make sense of negative cross ratios if we allow overlapping (overwinding). Is it possible to make geometric sense of the cross ratio as a complex number ?

(viii) Study the space of "triangulations" of Σ_g, which lift to honest triangulations of $\widetilde{\Sigma}_g$. In particular, it would be interesting to generate a complex from the triangulations, similar to the complex of curves studied by Harvey in [Har92].

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