Due Date: July 9, 2008
Legend: (−) easy; (+) hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 11.1 Let G be a bipartite graph with partite sets A and B. Prove that for all integers $k < |A|$ and $\ell < |B|$ it holds that

$$d(A,B) = \frac{1}{\binom{|A|}{k}\binom{|B|}{\ell}} \sum_{X \in \binom{A}{k}} \sum_{Y \in \binom{B}{\ell}} d(X,Y).$$

Remind that $\binom{A}{k}$ denotes the set of subsets of A of size k, and $\binom{B}{\ell}$ the set of subsets of B of size ℓ.

Exercise 11.2

1. Prove the following. For every natural number $r \geq 2$ and every real number $d \in (0,1)$ there exist $\varepsilon_0 < 1$ and $c > 0$ such that for all $\varepsilon \leq \varepsilon_0$: if G is an r-partite graph with its partite sets $V_1, \dots, V_r$, where $|V_1| = \cdots = |V_r| = n$ and (V_i, V_j) is ε-regular with density d for all $1 \leq i < j \leq r$, then G contains at least cn^r copies of K_r.

2. Prove the following. For every natural number $r \geq 2$ and every positive real number $\delta > 0$ there exist a natural number n_0 and a positive real number $c > 0$ such that for every graph G satisfying $n(G) \geq n_0$ and $e(G) \geq \left(1 + \frac{1}{r-1} + \delta\right)\binom{n}{2}$ contains at least cn^r copies of K_r.

Exercise 11.3 (−) Show that an ε-regular partition $\{V_0, V_1, \dots, V_k\}$ of a graph G is also an ε-regular partition of its complement $\overline{G}$.

Exercise 11.4 (+) Prove that for every positive number $c > 0$ there exists a positive number c' such that if an n-vertex graph G has more than cn^2 edges, then G contains a regular subgraph with $c'n^2$ edges. (Hint: Use Hall's theorem repeatedly.)

Exercise 11.5 (+) Prove that for every $\delta > 0$ there exists a natural number n_0 and a positive real number $c > 0$ such that for every graph $G = (V, E)$ with $n(G) \geq n_0$ that contains at least cn^3 K_3's there exists a set $F \subseteq E$ of edges such that $|F| \leq \delta\binom{n}{2}$ and $G - F$ contains no K_3.

Exercise 11.6 Use the exercise just above, prove the following: For any $\varepsilon > 0$ there exists an integer N such that for any $n \geq N$ and $S \subseteq \{1, \dots, n\}$, $|S| \geq \varepsilon n$, there exists a three-element arithmetic progression in S. (Hint: Create the following 3-partite graph from S. The first partite set A is $\{1, \dots, n\}$, the second partite set B is $\{1, \dots, 2n\}$, and the third partite set C is $\{1, \dots, 3n\}$. Two vertices $a \in A$ and $b \in B$ are adjacent if $b - a \in S$; Two vertices $b \in B$ and $c \in C$ are adjacent if $c - b \in S$; Two vertices $a \in A$ and $c \in C$ are adjacent if $(c - a)/2 \in S$.)