

## Today's contents

- Random graphs
- Probabilistic methods
- The Galton-Watson process

## Topics on Computing and Mathematical Sciences I Graph Theory (10) Random Graphs

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## Probability space

## Definition (Probability space)

A pair  $(\Omega, \mathbb{P})$  is a **finite probability space** if  $\Omega$  is a finite set and  $\mathbb{P}$  is a map from  $\Omega$  into  $\mathbb{R}_{\geq 0}$  with  $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Example: An unbiased dice

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\mathbb{P}(1) = \mathbb{P}(2) = \mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6) = \frac{1}{6}$$



## Random graphs

$V = \{1, \dots, n\}$  a set of vertices

$$N = \binom{n}{2} = \frac{n(n-1)}{2}$$

$p$  a number with  $0 \leq p \leq 1$

## Definition (Random graph I)

- We select the edges of  $K_n$  independently, with probability  $p$
- $\mathcal{G}(n, p) = (\mathcal{G}_n, \mathbb{P}_p)$  is a probability space, where  $\mathcal{G}_n$  is the set of all  $2^N$  graphs on  $V$  and

$$\mathbb{P}_p(H) = p^m(1-p)^{N-m}$$

if the graph  $H$  on  $V$  has precisely  $m$  edges

- $G_{n,p}$  denotes a random graph in the space  $\mathcal{G}(n, p)$

## Random graphs

$V = \{1, \dots, n\}$  a set of vertices

$$N = \binom{n}{2} = \frac{n(n-1)}{2}$$

$M$  an integer with  $0 \leq M \leq N$

## Definition (Random Graph II)

- $\mathcal{G}(n, M) = (\mathcal{G}_{n,M}, \mathbb{P}_M)$  is a probability space, where  $\mathcal{G}_{n,M}$  is the set of all  $\binom{N}{M}$  subgraphs of  $K_n$  with  $M$  edges and

$$\mathbb{P}_M(H) = \binom{N}{M}^{-1}$$

for all  $H \in \mathcal{G}_{n,M}$

- $G_{n,M}$  denotes a random graph in the space  $\mathcal{G}(n, M)$

## Tools from probability: Expectation

$(\Omega, \mathbb{P}) = \mathcal{G}(n, p)$  or  $\mathcal{G}(n, M)$

A **random variable**  $X$  on  $\Omega$  is a mapping  $X : \Omega \rightarrow \mathbb{R}$

The **expectation**  $\mathbb{E}(X)$  of  $X$  is  $\mathbb{E}(X) = \sum_{G \in \Omega} \mathbb{P}(G) \cdot X(G)$

## Lemma 10.1

For two random variables  $X$  and  $Y$  on  $\Omega$ ,  $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

## Proof

$$\begin{aligned} \mathbb{E}(X + Y) &= \sum_{G \in \Omega} \mathbb{P}(G) \cdot (X(G) + Y(G)) \\ &= \sum_{G \in \Omega} \mathbb{P}(G) \cdot X(G) + \sum_{G \in \Omega} \mathbb{P}(G) \cdot Y(G) = \mathbb{E}(X) + \mathbb{E}(Y) \end{aligned} \quad \square$$

## Tools from probability: Markov's inequality

$(\Omega, \mathbb{P}) = \mathcal{G}(n, p)$  or  $\mathcal{G}(n, M)$

$X = X(G)$  a non-negative random variable on  $\Omega$

$a > 0$  a number

## Lemma 10.2 (Markov's inequality)

$$\text{Prob}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

## Proof

$$\begin{aligned} \mathbb{E}(X) &= \sum_{G \in \Omega} \mathbb{P}(G) \cdot X(G) \geq \sum_{\substack{G \in \Omega \\ X(G) \geq a}} \mathbb{P}(G) \cdot X(G) \\ &\geq \sum_{\substack{G \in \Omega \\ X(G) \geq a}} \mathbb{P}(G) \cdot a = \text{Prob}(X \geq a) \cdot a \end{aligned} \quad \square$$

## Structure of random graphs

What does  $G_{n,p}$  look like?

For further information, refer to "*Random Graphs*" by Bollobás (1985)

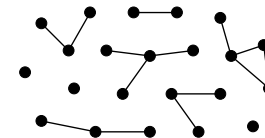
An isolated tree is a connected component without cycles

Denote by  $T(G)$  the number of vertices contained in isolated trees of  $G$

Clearly  $T(G) \leq n$

Suppose  $p = c/n$ , where  $c$  is a constant with  $0 < c < 1$ ; Then

- $\mathbb{E}(T(G_{n,p})) = n + O(1)$
- For almost every  $G_{n,p}$ , the size of the largest component is  $O(\ln n)$



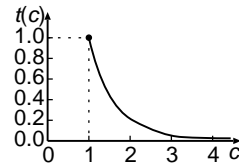
## Structure of random graphs

$T(G)$  # of vertices contained in isolated trees of  $G$

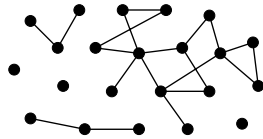
Suppose  $p = c/n$ , where  $c$  is a constant with  $c > 1$ ; Then

- $\mathbb{E}(T(G_{n,p})) = t(c) \cdot n + O(1)$ , where

$$t(c) = \frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (c \cdot e^{-c})^k$$



- For almost every  $G_{n,p}$ , it has a unique giant component, and all other vertices form trees of size  $O(\ln n)$



## Structure of random graphs

The following is the most celebrated theorem in the theory of random graphs:

Erdős-Rényi theorem (1960)

Suppose  $p = c \cdot \frac{\ln n}{n}$ . Then

$$\lim_{n \rightarrow +\infty} \text{Prob}(G_{n,p} \text{ is connected}) = \begin{cases} 1 & \text{if } c > 1 \\ 0 & \text{if } 0 < c < 1 \end{cases}$$

In other words,  $\frac{\ln n}{n}$  is a sharp threshold for the connectivity of  $G_{n,p}$

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## A theorem of Erdős

Theorem 10.3 (Erdős '59) (This fact was stated in Lecture 7)

For every  $k \geq 2$ , there exists a graph  $G$  with  $\chi(G) > k$  and  $g(G) > k$ .

Reminder:

$\chi(G)$  = the chromatic number of  $G$

$g(G)$  = the girth (the length of a shortest cycle) of  $G$

Let us see that this theorem can be proved using random graphs

This approach is called a **probabilistic method**

Proof outline of the theorem of Erdős

Proof outline of Theorem 10.3

Choose  $p := n^{-\frac{k}{k+1}} = \frac{1}{n} \cdot n^{\frac{1}{k+1}}$  ( $> \frac{1}{n}$ , and  $> \frac{\ln n}{n}$  for sufficiently large  $n$ ) and consider  $\mathcal{G}(n, p)$  for all  $n$

We have that as  $n$  goes to infinity

- $\alpha(G_{n,p}) < \frac{n}{2k}$  holds with high probability (Lemma 10.4)
- # of all cycles of length  $\leq k$  of  $G_{n,p}$  is at most  $\frac{n}{2}$  with high probability (Lemma 10.5)

As a result, there must exist a graph with desired properties

Reminder:

$\alpha(G)$  = the size of a maximum independent set of  $G$   
 $\chi(G) \cdot \alpha(G) \geq n$

Properties of random graphs

Lemma 10.4

If  $p := n^{-\frac{k}{k+1}}$ ,  $\exists n_1 \in \mathbb{N}$  such that for all  $n \geq n_1$

$$\text{Prob}\left(\alpha(G_{n,p}) \geq \frac{n}{2k}\right) < \frac{1}{2}$$

Lemma 10.5

If  $p := n^{-\frac{k}{k+1}}$ ,  $\exists n_2$  such that for all  $n \geq n_2$

$$\text{Prob}\left(\# \text{ of all cycles of length } \leq k \text{ of } G_{n,p} \geq \frac{n}{2}\right) < \frac{1}{2}$$

By Lemmas 10.4 and 10.5, there exists a graph  $G$  on  $V$  with

- $\alpha(G) < \frac{n}{2k}$
- fewer than  $\frac{n}{2}$  cycles of length  $\leq k$ .

Properties of random graphs

Proof of Lemma 10.4

$p := n^{-\frac{k}{k+1}}$ ; Suppose  $2 \leq r := \lceil \frac{n}{2k} \rceil$

- $\text{Prob}(\text{a fixed } r\text{-set } \subseteq V \text{ is independent}) = (1-p)^{\binom{r}{2}}$
- $\text{Prob}(\alpha(G_p) \geq r) \leq \binom{n}{r} (1-p)^{\binom{r}{2}}$   
 $\leq n^r (1-p)^{r(r-1)/2} = (n(1-p)^{(r-1)/2})^r$   
 $\leq (ne^{-p(r-1)/2})^r \quad (1-p \leq e^{-p})$   
 $\leq (ne^{-pr/2} \cdot e^{\frac{1}{2}})^r \quad (p \leq 1)$
- $pr \geq \frac{n}{2k} \cdot n^{-\frac{k}{k+1}} = \frac{1}{2k} \cdot n^{\frac{1}{k+1}} \geq 3 \ln n$  for all  $n \geq n'_1$
- $ne^{-pr/2} \cdot e^{\frac{1}{2}} \leq ne^{-3 \ln n / 2} \cdot e^{\frac{1}{2}} = \left(\frac{e}{n}\right)^{1/2}$  for all  $n \geq n'_1$
- $\text{Prob}(\alpha(G_p) \geq r) \leq \left(\frac{e}{n}\right)^{r/2}$  for all  $n \geq n'_1$

$\frac{e}{n}$  converges to 0 as  $n$  goes to  $+\infty$  □

Properties of random graphs

Proof of Lemma 10.5

$i \in \mathbb{N}$  an integer with  $3 \leq i \leq k$

- (# of possible  $i$ -cycles on  $V$ ) =  $\binom{n}{i} \frac{(i-1)!}{2}$
- Every such cycle  $C$  appears with probability  $p^i$

$X$  a random variable that counts # of all cycles of length  $\leq k$  of  $G_{n,p}$

- $\mathbb{E}(X) = \sum_{i=3}^k \binom{n}{i} \frac{(i-1)!}{2} p^i \leq \frac{1}{2} \sum_{i=3}^k n^i p^i \leq \frac{1}{2} (k-2) n^k p^k$
- $\text{Prob}(X \geq \frac{n}{2}) \leq \frac{\mathbb{E}(X)}{n/2}$  (Markov's inequality)  
 $\leq (k-2) \frac{(np)^k}{n}$  ( $p = n^{-\frac{k}{k+1}}$ )  
 $= (k-2) n^{-\frac{1}{k+1}}$

$n^{-\frac{1}{k+1}}$  converges to 0 as  $n$  goes to  $+\infty$  □

Proof of the theorem of Erdős

Proof of Theorem 10.3

- By Lemmas 10.4 and 10.5, there exists a graph  $G$  on  $V$  with  $\alpha(G) < \frac{n}{2k}$  and fewer than  $\frac{n}{2}$  cycles of length  $\leq k$ .
- Delete one vertex from each of cycles of length  $\leq k$   
Let  $H$  be the resulting graph
- We have  $g(H) > k$
- $\alpha(H) \leq \alpha(G) < \frac{n}{2k}$  and  $\chi(H) \cdot \alpha(H) \geq n(H) \geq \frac{n}{2}$   
 $\Rightarrow \chi(H) > k$
- Thus,  $H$  is a graph with desired properties □

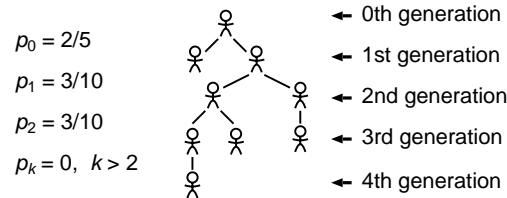
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The Galton-Watson process

Definition (The Galton-Watson process  $\{Z_i : i = 0, 1, \dots\}$  (around 1873))

- $Z_i$  denotes the number of people in the  $i$ -th generation
- $Z_0 = 1$  (There exists a unique root)
- $Z_1$  has a fixed distribution:  $\text{Prob}(Z_1 = k) = p_k, k \geq 0$
- Each child produces offspring according to the same distribution

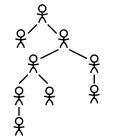


Probability distribution  $\{p_k : k \geq 0\}$  is called an **offspring distribution**

The Galton-Watson process

$\{p_k : k \geq 0\}$  an offspring distribution

$$\text{Prob}(\text{extinction}) = \text{Prob}(Z_k = 0 \text{ for some } k \geq 0)$$



We want to determine whether **Prob(extinction) = 1 or not**

→ Basically, it only depends on  $\mu := \mathbb{E}(Z_1) = \sum_{k=0}^{+\infty} k \cdot p_k$

Suppose  $p_0 + p_1 = 1$ ; Clearly,  $\mu = p_1$

If  $p_1 < 1$ , then  $\text{Prob}(\text{extinction}) = 1$

If  $p_1 = 1$ , then  $\text{Prob}(\text{extinction}) = 0$

Thus, it suffices to consider the case where  $p_0 + p_1 < 1$

## The Galton-Watson process

$\{p_k : k \geq 0\}$  an offspring distribution

## Theorem 10.6

Suppose  $p_0 + p_1 < 1$ . Then

$$\text{Prob}(\text{extinction}) \begin{cases} = 1 & \text{if } \mu \leq 1 \\ < 1 & \text{if } \mu > 1, \end{cases}$$

where  $\mu = \mathbb{E}(Z_1) = \sum_{k=0}^{+\infty} k \cdot p_k$

Note that the population becomes extinct with probability one even if  $\mu = 1$

## The Galton-Watson process

## Intuitive explanation of Theorem 10.6 (i)

$q = \text{Prob}(\text{extinction})$ ; We have  $0 \leq q \leq 1$

- $\text{Prob}(\text{extinction} | Z_1 = k) = q^k$
- $q = \sum_{k=0}^{\infty} p_k \cdot \text{Prob}(\text{extinction} | Z_1 = k) = \sum_{k=0}^{\infty} p_k q^k$

Define  $f(s) = \sum_{k=0}^{\infty} p_k s^k$  ( $s \in \mathbb{R}$ )

- $f(q) = q$  and  $f(1) = 1$
- $f'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}$  and  $f'(1) = \sum_{k=1}^{\infty} k p_k = \mu$
- $s^k$  is strictly convex on  $[0, 1]$  if  $k \geq 2$ , and  $p_k > 0$  for some  $k \geq 2$   
 $\Rightarrow f(s)$  is strictly convex on  $[0, 1]$

## The Galton-Watson process

## Intuitive explanation of Theorem 10.6 (ii)

Suppose  $f'(1) = \mu \leq 1$

- Equation  $f(s) = s$  has a unique solution  $s = 1$  on  $[0, 1]$
- Thus, we obtain  $q = 1$

Suppose  $f'(1) = \mu > 1$ ; Remark that  $f(0) = p_0 \geq 0$

- Equation  $f(s) = s$  has two distinct solutions  $s = 1$  and  $\alpha \in [0, 1)$
- We can expect that  $q = \alpha$  □

