Topics on Computing and Mathematical Sciences I Graph Theory (9) Extremal Graph Theory I

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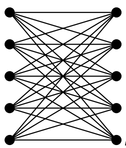
Today's contents

1. Complete multipartite graphs, Turán graphs
2. Turán's theorem
3. Turán-type problems and Erdős-Simonovits-Stone theorem
4. Open problems

Mantel's theorem (recap)

Theorem 2.7 (Extremality for having no K_3, Mantel 1907)

The maximum number of edges in an n-vertex graph that contains no K_3 is $\lfloor n^2/4 \rfloor$



$K_{5,5}$

Promised to answer...

There, we consider graphs containing no K_3

Questions

- What about graphs containing no K_4
- What about graphs containing no K_r (r fixed)
- What about graphs containing no $K_{r,r}$
- What about graphs containing no Petersen graph
- ...

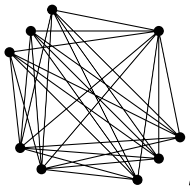
We will answer these questions (completely or partially) in today's class

Complete multipartite graphs

$G = (V, E)$ a graph; r a natural number

Definition (Complete r-partite graph)

G is complete r-partite if $\exists$ a partition $V_1 \cup \cdots \cup V_r$ of V s.t. $\{u, v\} \in E \Leftrightarrow \{u, v\} \not\subseteq V_i$ for any i; Denoted by $K_{n_1,\ldots,n_r}$ if $|V_i| = n_i$



$K_{1,2,3,3}$

Remark

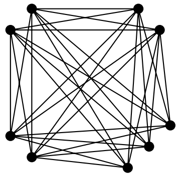
G r-partite $\Rightarrow$ G contains no K_{r+1}

Turán graphs

n, r natural numbers

Definition (Turán graph)

A Turán graph $T_{n,r}$ is an n-vertex complete r-partite graph with the sizes of its partite sets as equal as possible (the partite sets have $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$ vertices)



$T_{9,4} \simeq K_{2,2,2,3}$

Intuition (?)

$T_{n,r}$ maximizes # edges among all graphs containing no K_{r+1} (?)

Turán graphs are extremal among the r-partite graphs

n, r natural numbers

Lemma 9.1 (Extremality for r-partiteness)

The Turán graph $T_{n,r}$ is a unique n-vertex r-partite graph that has a maximum number of edges

Proof idea.

Let G be an n-vtx r-partite graph that has a max # of edges

- G must be complete r-partite; $G \simeq K_{n_1,\ldots,n_r}$
- $e(G) = \binom{n}{2} - \sum_{i=1}^{r} \binom{n_i}{2}$ and this is maximized when $n_1, \ldots, n_r$ are as equal as possible □

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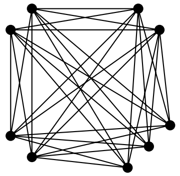
Turán's theorem

n, r natural numbers

Theorem 9.2 (Extremality for having no K_r; Turán '41)

The maximum number of edges in an n-vertex graph that contains no K_{r+1} is $e(T_{n,r})$



$T_{9,4} \simeq K_{2,2,2,3}$

Easy to see: this max $\# \geq e(T_{n,r})$

Proof of Turán's theorem

Proof idea.

Induction on r; Easy when $r = 1$ so suppose $r \geq 2$

- Let $G = (V, E)$ be an n-vertex graph that contains no K_{r+1} and has a maximum number of edges
- $v \in V$ a vertex of max degree ($d_G(v) = \Delta(G)$)
- Count the edges in $G[N(v)]$, this is at most $e(T_{\Delta(G),r-1})$ (why?)
- Count the remaining edges, this is at most $\Delta(G)(n - \Delta(G))$
- $\therefore e(G) \leq e(T_{\Delta(G),r-1}) + \Delta(G)(n - \Delta(G))$
- On the other hand, $\exists$ an n-vertex complete r-partite graph with $e(T_{\Delta(G),r-1}) + \Delta(G)(n - \Delta(G))$ edges
- By Lem 9.1, $e(T_{\Delta(G),r-1}) + \Delta(G)(n - \Delta(G)) \leq e(T_{n,r})$ □

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The Turán-type problem

Question (Turán-type problem)

Given a natural number n and a graph H,
What is the maximum number of edges in an n-vertex graph that contains no H?

Notation

$\mathrm{ex}(n, H) = \max\{e(G) \mid n(G) = n, G \not\supseteq H\}$

We saw...

- $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ (Mantel)
- $\mathrm{ex}(n, K_{r+1}) = e(T_{n,r})$ (Turán)

For other graphs??

Erdős-Simonovits-Stone theorem

A complete answer for the Turán-type problems

Theorem 9.3 (Erdős, Simonovits '66; Erdős-Simonovits-Stone thm)

$\forall$ graph H

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H)-1};$$

In other words,

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + \mathrm{o}(n^2)$$

The chromatic number answers the Turán-type problem

Proof outline of Erdős-Simonovits-Stone theorem

Proof outline I

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm (in the next slide)
2. Deduce the Erdős-Stone thm from its weaker version
3. Prove the weaker version of Erdős-Stone thm

Proof outline II

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm (in the next slide)
2. Use Szemerédi's regularity lemma to prove the embedding lemma (next lecture)
3. Use the embedding lemma to prove Erdős-Stone thm

Erdős-Stone theorem

Theorem 9.4 (Erdős, Stone '46)

$\forall\ s \geq 1, r \geq 2$ natural numbers

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,T_{rs,r})}{\binom{n}{2}} = 1 - \frac{1}{r-1}$$

Consequence of Erdős-Stone thm

$\mathrm{ex}(n,H) \approx (1-\frac{1}{r-1})\binom{n}{2} \approx \mathrm{ex}(n,T_{rs,r})$ where $r = \chi(H)$

Deducing Erdős-Simonovits-Stone from Erdős-Stone

Proof idea of Thm 9.3 using Thm 9.4

Squeeze theorem from Calculus; Let $r = \chi(H)$

- $\exists\ t \in \mathbb{N}$: $\mathrm{ex}(n,H) \leq \mathrm{ex}(n,T_{rt,r})$ $(\because H \subseteq T_{rn(H),r})$
- $\displaystyle\lim_{n\to\infty} \frac{\mathrm{ex}(n,T_{rt,r})}{\binom{n}{2}} = 1 - \frac{1}{r-1}$ (Thm 9.4)
- $e(T_{n,r-1}) \leq \mathrm{ex}(n,H)$ $(\because \chi(T_{n,r-1}) < \chi(H)$ (so $H \not\subseteq T_{n,r-1}$))
- $\therefore$ Remains to show $\displaystyle\lim_{n\to\infty} \frac{e(T_{n,r-1})}{\binom{n}{2}} = 1 - \frac{1}{r-1}$

Some calculation to finish the proof (1/2)

$$\begin{aligned} e(T_{n,r-1}) &\leq \binom{n}{2} - (r-1)\binom{\lfloor n/(r-1)\rfloor}{2} \\ &= \binom{n}{2} - (r-1)\frac{1}{2}\left\lfloor \frac{n}{r-1}\right\rfloor \left(\left\lfloor \frac{n}{r-1}\right\rfloor - 1\right) \\ &\leq \binom{n}{2} - \frac{r-1}{2}\left(\frac{n}{r-1} - 1\right)\left(\frac{n}{r-1} - 2\right) \\ &= \binom{n}{2} - \frac{r-1}{2}\left(\frac{n^2}{(r-1)^2} - \frac{3n}{r-1} + 2\right) \\ &= \binom{n}{2} - \frac{n^2}{2(r-1)} - \frac{3n}{2} + (r-1) \end{aligned}$$

$$\therefore \lim_{n\to\infty} \frac{\mathrm{ex}(T_{n,r-1})}{\binom{n}{2}} \leq 1 - \frac{1}{r-1}$$

Some calculation to finish the proof (2/2)

$$\begin{aligned} e(T_{n,r-1}) &\geq \binom{n}{2} - (r-1)\binom{\lceil n/(r-1)\rceil}{2} \\ &= \binom{n}{2} - (r-1)\frac{1}{2}\left\lceil \frac{n}{r-1}\right\rceil \left(\left\lceil \frac{n}{r-1}\right\rceil - 1\right) \\ &\geq \binom{n}{2} - \frac{r-1}{2}\frac{n}{r-1}\left(\frac{n}{r-1} - 1\right) \\ &= \binom{n}{2} - \frac{n^2}{2(r-1)} + \frac{n}{2} \end{aligned}$$

$$\therefore \lim_{n\to\infty} \frac{\mathrm{ex}(T_{n,r-1})}{\binom{n}{2}} \geq 1 - \frac{1}{r-1} \qquad \square$$

Proof outline of Erdős-Simonovits-Stone theorem

Proof outline I

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm
2. Deduce the Erdős-Stone thm from its weaker version
3. Prove the weaker version of Erdős-Stone thm

Proof outline II

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm
2. Use Szemerédi's regularity lemma to prove the embedding lemma (next lecture)
3. Use the embedding lemma to prove Erdős-Stone thm

Erdős-Stone theorem, a weaker version

Rephrasing the Erdős-Stone thm

$\forall\ s \geq 1, r \geq 2$ natural numbers $\forall\ \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \forall\ G$: $n(G) = n \geq n_0$ and

$$e(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2} \Rightarrow G \supseteq T_{rs,r}$$

Lemma 9.5 (Weaker version of the Erdős-Stone thm)

$\forall\ s \geq 1, r \geq 2$ natural numbers $\forall\ \varepsilon > 0\ \exists n_1 \in \mathbb{N}\ \forall n \geq n_1\ \forall\ G$: $n(G) = n \geq n_1$ and

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right) n \Rightarrow G \supseteq T_{rs,r}$$

Note: $\delta(G) \geq cn \Rightarrow e(G) \geq cn^2/2 \geq c\binom{n}{2}$

From the weaker version to Erdős-Stone: Proof idea (1/4)

Given s, r, ε as in Erdős-Stone

- We order the vertices $v_1, \dots, v_n$ of G s.t. v_i minimizes the deg of $G_i = G - \{v_1, \dots, v_{i-1}\}$ (Note: $G_1 = G$)
- Namely $d_{G_i}(v_i) = \delta(G_i)$
- Assume for some i: $\delta(G_{i-1}) < (1 - \frac{1}{r-1} + \frac{\varepsilon}{2})n(G_{i-1})$ and $\delta(G_i) \geq (1 - \frac{1}{r-1} + \frac{\varepsilon}{2})n(G_i)$
- Claim: Can determine $n_0 = n_0(s, r, \varepsilon)$ s.t. $n(G_i) \geq n_1(s, r, \varepsilon/2)$
  - Then, $G_i \supseteq T_{rs,r}$ by the weak version

From the weaker version to Erdős-Stone: Proof idea (2/4)

- By the assumption $e(G) \geq \left(1 - \dfrac{1}{r-1} + \varepsilon\right)\dbinom{n}{2}$
- On the other hand

$$\begin{aligned}
e(G) &= e(G_i) + \sum_{j=1}^{i-1} d_{G_j}(v_j) \\
&< \binom{n-i+1}{2} + \sum_{j=1}^{i-1}\left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right)(n-j+1) \\
&= \binom{n-i+1}{2} + \left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right)\left((n+1)(i-1) - \frac{(i-1)i}{2}\right) \\
&= \binom{n-i+1}{2} + \left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right)\left(\binom{n}{2} - \binom{n-i+1}{2} + i - 1\right)
\end{aligned}$$

From the weaker version to Erdős-Stone: Proof idea (3/4)

- Putting together, we get

$$\begin{aligned}
\frac{\varepsilon}{2}\binom{n}{2} &< \left(\frac{1}{r-1} - \frac{\varepsilon}{2}\right)\binom{n-i+1}{2} + \left(1 - \frac{1}{r-1} + \frac{\varepsilon}{2}\right)(i-1) \\
\binom{n-i+1}{2} &> \frac{\frac{\varepsilon}{2}\binom{n}{2} - (1 - \frac{1}{r-1} + \frac{\varepsilon}{2})(i-1)}{\frac{1}{r-1} - \frac{\varepsilon}{2}} \\
&\geq \frac{\frac{\varepsilon}{2}\binom{n}{2} - (1 - \frac{1}{r-1} + \frac{\varepsilon}{2})n}{\frac{1}{r-1} - \frac{\varepsilon}{2}}
\end{aligned}$$

- We want to determine $n_0(s, r, \varepsilon)$ s.t. $\forall\ n \geq n_0(s, r, \varepsilon)$ it holds

$$\frac{\frac{\varepsilon}{2}\binom{n}{2} - (1 - \frac{1}{r-1} + \frac{\varepsilon}{2})n}{\frac{1}{r-1} - \frac{\varepsilon}{2}} \geq \binom{n_1(s, r, \varepsilon/2)}{2}$$

From the weaker version to Erdős-Stone: Proof idea (4/4)

- For brevity, write the LHS as $c_1(r, \varepsilon)n^2 - c_2(r, \varepsilon)n$ for some c_1, c_2
- To have $c_1(r, \varepsilon)n^2 - c_2(r, \varepsilon)n \geq \binom{n_1(s,r,\varepsilon/2)}{2}$, it suffices to have

$$n \geq \frac{c_2(r, \varepsilon) + \sqrt{c_2(r, \varepsilon)^2 + 4c_1(r, \varepsilon)\binom{n_1(s,r,\varepsilon/2)}{2}}}{2c_1(r, \varepsilon)}$$

- Choose this RHS as $n_0(s, r, \varepsilon)$
- Then, $n(G_i) = n - i + 1 \geq n_1(s, r, \varepsilon/2)$ □

Proof outline of Erdős-Stone-Simonovits theorem

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2. Deduce the Erdős-Stone thm from its weaker version
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Proof outline II

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm
2. Use Szemerédi's regularity lemma to prove the embedding lemma (next lecture)
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A weaker version of the Erdős-Stone theorem

Lemma 9.5 (Weaker version of the Erdős-Stone thm)

$\forall\ s \geq 1, r \geq 2$ natural numbers $\forall\ \varepsilon > 0\ \exists n_1 \in \mathbb{N}\ \forall n \geq n_1\ \forall\ G$: $n(G) = n \geq n_1$ and

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right) n \Rightarrow G \supseteq T_{rs,r}$$

Proof outline

1. Induction on r
2. When $r = 2$, $\mathrm{ex}(n, T_{2s,2}) = \mathrm{O}(n^{1.5})$ so Lem holds (Exercise)
3. When $r > 2$, by induction we find $T_{(r-1)t,r-1}$, for some t ($> s$), in our graph G
4. Extend this $T_{(r-1)t,r-1}$ to $T_{rs,r}$ in G

Proof of the weaker version (1/3)

$s \geq 1, r \geq 3, \varepsilon > 0$ given, $n_1 = n_1(s, r, \varepsilon)$ to be determined later

- Assume G has $n \geq n_1(s, r, \varepsilon)$ vertices and satisfies the assumption in the lemma
- Let $t = \lceil s/\varepsilon \rceil$
- Assuming $n \geq n_1(t, r-1, \varepsilon)$, $G \supseteq T_{(r-1)t,r-1}$ (Induction)
- Let $V_1, \ldots, V_{r-1}$ the partite sets of $T_{(r-1)t,r-1}$
- A vtx $v \in V(G) \setminus (V_1 \cup \cdots V_{r-1})$ good if $|N_G(v) \cap V_i| \geq s$ for all $i \in \{1, \ldots, r-1\}$
- $g = \#$ good vertices of G
- (Good vertices are eligible to extend $T_{(r-1)t,r-1}$ to $T_{rs,r}$)

Proof of the weaker version (2/3)

- Double-count the non-edge pair of vertices between $V(G) \setminus (V_1 \cup \cdots \cup V_{r-1})$ and $V_1 \cup \cdots \cup V_{r-1}$
- From $V_1 \cup \cdots \cup V_{r-1}$:
  - Each vtx is non-adjacent to at most $(\frac{1}{r-1} - \varepsilon)n$ vtx's (Assumption on the min degree)
  - $\therefore$ This number $\leq (r-1)t(\frac{1}{r-1} - \varepsilon)n = t(1-(r-1)\varepsilon)n$
- From $V(G) \setminus (V_1 \cup \cdots \cup V_{r-1})$:
  - Each non-good vtx is non-adjacent to more than $t-s$ vertices in at least one V_i
  - This number $> (n-(r-1)t-g)(t-s) \geq (n-(r-1)t-g)s\frac{1-\varepsilon}{\varepsilon}$
- $\therefore (n-(r-1)t-g)s\frac{1-\varepsilon}{\varepsilon} < t(1-(r-1)\varepsilon)n$
- $\therefore g > 1-\dfrac{\varepsilon t(1-(r-1)\varepsilon)}{s(1-\varepsilon)}n - (r-1)t$

Proof of the weaker version (3/3)

- Assuming $n_1(s,r,\varepsilon) \geq \frac{\binom{t}{s}^{r-1}(s-1)+(r-1)t}{1-\frac{\varepsilon t(1-(r-1)\varepsilon)}{s(1-\varepsilon)}}$, we have

$$g > \left(1-\frac{\varepsilon t(1-(r-1)\varepsilon)}{s(1-\varepsilon)}\right) n_1 - (r-1)t \geq \binom{t}{s}^{r-1}(s-1)$$

- By the pigeonhole principle: $\exists$ good vertices $v_1,\ldots,v_s$ and $r-1$ sets $A_1,\ldots,A_{r-1}$ s.t.
  - $A_i \subseteq V_i$ and $|A_i| = s \ \forall\, i \in \{1,\ldots,r-1\}$
  - $|N_G(v_j) \cap A_i| = s \ \forall\, i \in \{1,\ldots,r-1\}, j \in \{1,\ldots,s\}$
- $G[A_1 \cup \cdots \cup A_{r-1} \cup \{v_1,\ldots,v_s\}]$ contains $T_{rs,r}$
- Checking the assumptions on $n_1(s,r,\varepsilon)$
  - $n_1(s,r,\varepsilon) \geq n_1(\lceil s/\varepsilon \rceil, r-1, \varepsilon)$
  - $n_1(s,r,\varepsilon) \geq \frac{\binom{\lceil s/\varepsilon\rceil}{s}^{r-1}(s-1)+(r-1)\lceil s/\varepsilon\rceil}{1-\frac{\varepsilon\lceil s/\varepsilon\rceil(1-(r-1)\varepsilon)}{s(1-\varepsilon)}}$
  - Not hard to find $n_1(s,r,\varepsilon)$ satisfying these two (detail omitted)

□

Proof outline of Erdős-Simonovits-Stone theorem

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Proof outline II (Next lecture)

1. Deduce the Erdős-Simonovits-Stone thm from the Erdős-Stone thm
2. Use Szemerédi's regularity lemma to prove the embedding lemma
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Reflection on the Erdős-Simonovits-Stone theorem

Erdős-Simonovits-Stone theorem

$\forall$ graph H

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H)-1};$$

In other words,

$$\mathrm{ex}(n,H) = \left(1-\frac{1}{\chi(H)-1}\right)\binom{n}{2} + \mathrm{o}(n^2)$$

The theorem does not tell anything when $\chi(H) = 2$ (i.e., H is bipartite); we just find $\mathrm{ex}(n,H) = \mathrm{o}(n^2)$

$\rightsquigarrow$ bipartite Turán-type problems

Extremality for complete bipartite graphs

Conjecture

$\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$ if $s \le t$

Known facts

- $\mathrm{ex}(n, K_{s,t}) = \mathrm{O}(n^{2-1/s})$ (Kővári, Sós, Turán '54; Exercise)
- $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-(s+t-2)/(st-1)})$ (Erdős, Spencer '74)
- True for $s = 2$ (Folklore; Exercise)
- True for $s = 3$ (Erdős, Rényi '62; Brown '66)
- True for $t \ge s! + 1$ (Kollár, Rónyai, Szabó '96)
- True for $t \ge (s-1)! + 1$ (Alon, Rónyai, Szabó '99)

Important to notice that these tight lower bounds highly depend on algebra (finite fields, algebraic geometry)

Extremality for even cycles

Conjecture

$\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+1/k})$

Known facts

- $\mathrm{ex}(n, C_{2k}) = \mathrm{O}(n^{1+1/k})$ (Bondy, Simonovits '74)
- $\mathrm{ex}(n, C_{2k}) = \Omega(n^{1+2/(3k-3)})$ (Lazebnik, Ustimenko, Woldar '95)
- True for $k = 2$ ($\because C_4 \simeq K_{2,2}$)
- True for $k = 3$ (Benson '66; Wenger '91)
- True for $k = 5$ (Benson '66; Wenger '91)

Extremality for trees: Erdős-Sós conjecture

Conjecture (Erdős, Sós '63)

T a tree with $k \ge 2$ edges $\Rightarrow \mathrm{ex}(n, T) \le \frac{1}{2}(k-1)n$

Known facts

- True for stars ($K_{1,k}$) (Easy; Exercise)
- True for paths (P_{k+1}) (Erdős, Gallai '59; Exercise)
- True for all trees w/ diameter ≤ 4 (McLennan '05)

Also refer to Exercise 3.4