Due Date: July 2, 2008
Legend: (−) easy; (+) hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 9.1

1. Prove that the Turán graph $T_{n,r}$ maximizes the sum of the squared degrees (i.e., $\sum_{v \in V(G)} d(v)^2$) among all n-vertex graphs G containing no K_{r+1}. (Hint: Mimic the proof of Turán's theorem)
2. Prove that the statement of the first part is no longer generally true if we consider maximizing $\sum_{v \in V(G)} d(v)^4$ instead.

Exercise 9.2 We are given a set of n distinct points in an equilateral triangle with side length 2 on the plane.

1. Prove that if $n = 5$, then there must exist a pair of points among these five points whose Euclidean distance is at most 1. (Hint: This is not much related to graphs, but not much to geometry either.)
2. Prove that the number of pairs of points among them whose Euclidean distance is greater than 1 is at most $\lfloor 3n(n-1)/8 \rfloor$. (Hint: Consider a graph with its vertices being the given n points and two vertices joined by an edge if and only if their Euclidean distance is greater than 1. What can you say about this graph using the first part of the exercise?)

Exercise 9.3 Let s, t, n be natural numbers such that $0 < s \leq t \leq n$.

1. Let G be an n-vertex graph which does not contain $K_{t,s}$. Prove that $\sum_{v \in V(G)} \binom{d(v)}{s} \leq (t-1)\binom{n}{s}$. (Hint: Count the number of copies of $K_{1,s}$ in two ways.)
2. Use the first part to prove that $\mathrm{ex}(n, K_{t,s}) \leq Cn^{2-1/s}$ for some constant C depending only on s and t. (Hint: Use the estimates $(\frac{a}{b})^b \leq \binom{a}{b} \leq a^b$ and Jensen's inequality.)

Exercise 9.4

1. Given n distinct points in the plane, prove that the distance is exactly one for at most $\mathrm{O}(n^{3/2})$ pairs.
2. Given n distinct points in the 3-dimensional space, prove that the distance is exactly 1 for at most $\mathrm{O}(n^{5/3})$ pairs.

Exercise 9.5 Let p be a prime number, and $\mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$. Consider the following graph G_p. The vertex set of G_p is $\mathbb{F}_p^2$, and an edge is drawn between $(a, b), (a', b') \in \mathbb{F}_p^2 \setminus \{(0, 0)\}$ if and only if $aa' + bb' = 1 \pmod p$.

1. Prove that G_p contains no $K_{2,2}$. (Hint: You can utilize the fact that $\mathbb{F}_p$ constitutes a field under the addition and the multiplication modulo p.)
2. Show that $e(G_p) \geq (p-1)(p^2-1)/2$.
3. Deduce that $\mathrm{ex}(n, K_{2,2}) = \Omega(n^{3/2})$.

Exercise 9.6 Erdős and Sós (1963) conjectured that if T is a tree with $k \geq 2$ edges then $\mathrm{ex}(n, T) \leq \frac{1}{2}(k-1)n$. This is still open.

1. (−) Verify the Erdős–Sós conjecture above when T is a star (i.e., $T = K_{1,k}$).
2. Verify the Erdős–Sós conjecture above when T is a path (i.e., $T = P_{k+1}$). (Hint: First prove that every connected graph G contains a path of length at least $\min\{2\delta(G), n(G) - 1\}$.)