Topics on Computing and Mathematical Sciences I Graph Theory (4) Matchings II

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April 30, 2008

"Last updated: Tue Jul 15 21:55 JST 2008"

Today's contents

- Matchings in bipartite graphs
- Independent sets
- Matchings in general graphs

Matchings (Recap)

We promised to answer the following questions

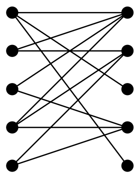
- Is it always possible to find a matching and a vertex cover of the same size for bipartite graphs?
  - Answer: Yes (Kőnig-Egerváry theorem)
- Is it possible to certify the non-existence of a perfect matching in a (non-bipartite) graph easily?
  - Answer: Yes (Tutte's theorem)

Matchings in bipartite graphs: Hall's theorem

First, a certificate for the non-existence of a perfect matching

Theorem 4.1 (Hall's theorem '35)

$G = (V, E)$ a bipartite graph with partite sets X, Y
G has a matching saturating $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$



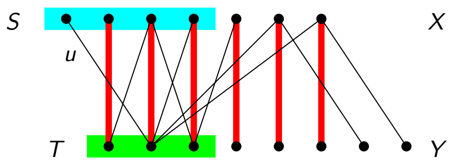
Note: Hall's theorem gives a good characterization for bipartite graphs with perfect matchings

Proof of Hall's theorem

Proof Idea.

[⇒] Easy direction
[⇐] Proof by Contradiction

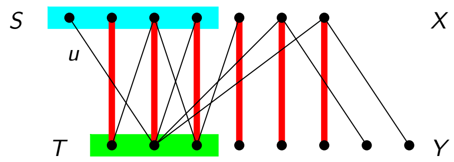
- M a maximum matching not saturating X
- We want: a set $S \subseteq X$ with $|N(S)| < |S|$
- Fix an M-unsaturated vertex $u \in X$
- $S = \{x \in X \mid \exists$ a u, x-path that is M-alternating$\}$
- $T = \{y \in Y \mid \exists$ a u, y-path that is M-alternating$\}$



Proof of Hall's theorem (continued)

Proof Idea (cont'd).

- Claim 1: M saturates T and $S \setminus \{u\}$
  (Consequence: $|T| = |S \setminus \{u\}| = |S| - 1 < |S|$)
- Claim 2: $T = N(S)$
  (Consequence: $|N(S)| = |T| < |S|$) □



Hall's theorem: Application

Theorem 4.2 (Frobenius '17)

Every k-regular bipartite graph has a perfect matching ($k \geq 1$)

Proof idea.

Double counting: count the number of edges between S and $N(S)$ to apply Hall's theorem (Also remember Exercise 1.3) □

Kőnig-Egerváry theorem: Strong duality

Definition (Matching number and covering number (recap))

$\alpha'(G)$ = the size of a maximum matching of G
$\beta(G)$ = the size of a minimum vertex cover of G

Corollary 3.12 (Weak duality, last lecture)

For any graph G, $\alpha'(G) \leq \beta(G)$

Theorem 4.3 (Kőnig-Egerváry Theorem, Kőnig '31, Egerváry '31)

For any bipartite graph G, $\alpha'(G) = \beta(G)$

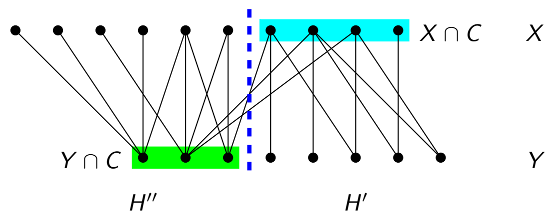
Proof of Kőnig-Egerváry Theorem

Setup: $G = (V, E)$ a bipartite graph with partite sets X, Y

Proof idea.

For bipartite graphs, we may use Hall's theorem

- Fix a minimum vertex cover $C \subseteq V$ arbitrarily
- Goal: Construct a matching $M \subseteq E$ s.t. $|M| = |C|$
- Let $H' = G[(X \cap C) \cup (Y \setminus C)]$ and $H'' = G[(X \setminus C) \cup (Y \cap C)]$

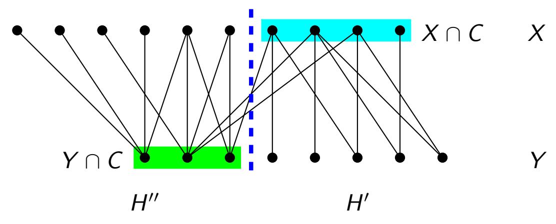


Proof of Kőnig-Egerváry Theorem (continued)

Setup: $G = (V, E)$ a bipartite graph with partite sets X, Y

Proof idea (cont'd).

- H' has a matching M' that saturates $X \cap C$ (why?)
- H'' has a matching M'' that saturates $Y \cap C$ (similarly)
- $M = M' \cup M''$ is a matching of G □



Remark: Weak duality and strong duality

For every bipartite graph G, we had two theorems

$\alpha'(G) \leq \beta(G)$ (weak duality)

If we find a matching M and a vertex cover C of the same size ($|M| = |C|$), then this *proves* M is a maximum matching and C is a minimum vertex cover

$\alpha'(G) = \beta(G)$ (strong duality, *min-max theorem*)

We can always find such M and C

This is similar to Duality Theorem of Linear Programming, and actually the Kőnig-Egerváry theorem can be derived from Duality of Linear Programming (via the integrality of the associated linear inequality system, keywords: total dual integrality, MFMC property)

Finding a maximum matching in a bipartite graph

Kőnig-Egerváry theorem suggests an algorithm to find a maximum matching of a bipartite graph G

Algorithm (Primal-Dual Method, rough idea)

1. Find a matching M and a vertex cover C of G
2. If $|M| = |C|$, they are optima
3. If not, we make either M larger or C smaller
4. Go back to 2nd step and repeat

This method is not that effective for the maximum matching problem for bipartite graphs, but is effective for other various problems, including the weighted maximum matching problem for bipartite graphs

Finding a maximum matching in a bipartite graph (2)

State of the art

- $O(\sqrt{n}m \log_n(n^2/m))$ (Feder, Motwani '95; Goldberg, Kennedy '97)
- $O(n^{2.376})$ randomized (Ibarra, Moran '81)

Open problem

Is it possible to get a *deterministic* $O(n^{2.376})$ algorithm for this problem, or faster?

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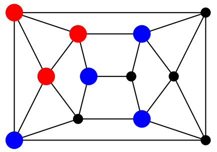
Independent sets, cliques

$G = (V, E)$ a graph; $S \subseteq V$ a vertex subset

Definition (Independent set, clique)

S is an independent set of G if no two vertices in S are adjacent
S is a clique of G if any two vertices in S are adjacent

A stable set is an alias of an independent set



Independence number and clique number

$G = (V, E)$ a graph; $S \subseteq V$ an independent set of G

Definition (maximum and maximal independent sets)

S is a maximum indep. set if $|S| \geq |S'|$ for all indep. sets S' of G;
S is a maximal indep. set if $S \cup \{v\}$ is not indep. for all $v \in V \setminus S$

Maximum and maximal cliques are defined similarly

Notation

$\alpha(G)$ = the size of a maximum independent set of G
(called the independence number or the stability number of G)
$\omega(G)$ = the size of a maximum clique of G
(called the clique number of G)

Relationship among independent sets, cliques, and matchings

Observations

- S an independent set of G $\Leftrightarrow$ S a clique of $\overline{G}$
- S an independent set of G $\Leftrightarrow$ $V \setminus S$ a vertex cover of G
- M a matching of G $\Leftrightarrow$ M an independent set of $L(G)$

In particular: $\alpha(G) + \beta(G) = n(G)$ and $\alpha(G) = \omega(\overline{G})$

α, β, ω

Karp '72

Finding a maximum independent set is NP-hard (Consequently, finding a minimum vertex cover and a maximum clique is NP-hard)

No polynomial-time algorithm is expected

Remark

A minimum vertex cover of a bipartite graph can be found in polynomial time (through Kőnig-Egerváry's theorem);
Consequently, a maximum independent set of a bipartite graph can be found in polynomial time

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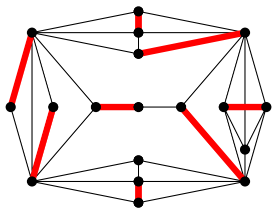
Perfect matchings in general graphs

Question

Is there a good way to certify the non-existence of a perfect matching in (not necessarily bipartite) graphs?



For bipartite graphs, we have Hall's theorem

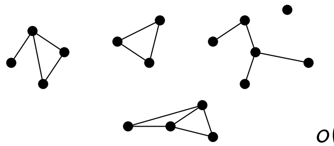
Odd components

Definition (Odd component)

An odd component of a graph G is a connected component with odd number of vertices

Notation

$o(G)$ = the number of odd components of G

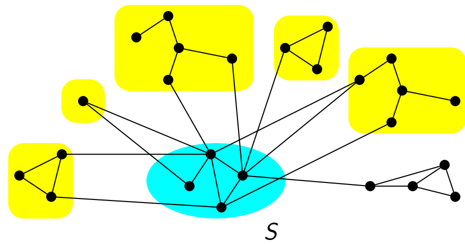


$o(G) = 3$

Odd components and a perfect matching

Observation

G has a perfect matching $\Rightarrow \forall\, S \subseteq V(G)$: $o(G{-}S) \leq |S|$



$|S| = 4$, $o(G{-}S) = 5$

In particular: $o(G) = 0$ $(= |\emptyset|)$

Consequence

$\exists\, S \subseteq V(G)$: $o(G{-}S) > |S| \Rightarrow G$ has no perfect matching

A characterization of graphs with a perfect matching: Tutte's theorem

Theorem 4.4 (Tutte '47)

G has a perfect matching $\Leftrightarrow \forall\, S \subseteq V(G)$: $o(G{-}S) \leq |S|$

Proof idea (Lovász '75).

[$\Rightarrow$] Already done

[$\Leftarrow$] Consider a maximal counterexample $G = (V, E)$, namely

1. G satisfies the RHS condition ($\forall\, S \subseteq V$: $o(G{-}S) \leq |S|$)
2. G has no perfect matching
3. $G{+}e$ has a perfect matching $\forall\, e \notin E$

- Let $U = \{v \in V \mid d_G(v) = n(G){-}1\}$; Consider $G{-}U$

Case 1: Every connected component of $G{-}U$ is complete

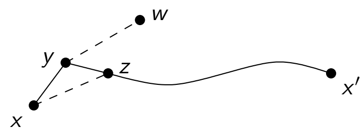
- We can find a perfect matching of G easily since $o(G{-}U) \leq |U|$

A characterization of graphs with a perfect matching: Tutte's theorem

Proof idea (continued).

Case 2: $\exists$ a connected component H of $G{-}U$ that's not complete

- H has two non-adjacent vertices x and x'
- The first three vertices x, y, z of a shortest x,x'-path satisfies $\{x, y\}, \{y, z\} \in E$ and $\{x, z\} \notin E$
- $\exists\, w \in V(G)$ s.t. $\{y, w\} \notin E$ (since $y \notin U$)
- Let M_1, M_2 perfect matchings of $G{+}\{x, z\}$, $G{+}\{y, w\}$ resp.

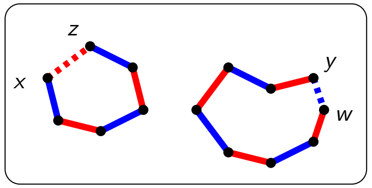


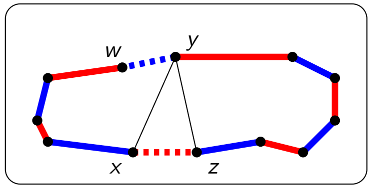
A characterization of graphs with a perfect matching: Tutte's theorem

Proof idea (further continued).

Claim: $M_1 \cup M_2$ contains a perfect matching of G

- Use the edges in $M_1 \cap M_2$, and look at $M_1 \triangle M_2$
- If $\{x, z\}$ and $\{y, w\}$ lie in different components of $M_1 \triangle M_2$...
- If $\{x, z\}$ and $\{y, w\}$ lie in the same component of $M_1 \triangle M_2$... □





A consequence of Tutte's theorem

Theorem 4.5 (Petersen 1891)

G 3-regular with no cut-edge $\Longrightarrow$ G has a perfect matching

Proof idea.

Apply Tutte's theorem

- Double counting: count the number of edges between S and the odd components of $G - S$ for an arbitrary $S \subseteq V(G)$ □

Another consequence: the Berge-Tutte formula

What about a min-max theorem for general graphs?

Theorem 4.6 (the Berge-Tutte formula; Berge '58)

$$\alpha'(G) = \min \left\{ \frac{1}{2}(n(G) + |S| - o(G - S)) \mid S \subseteq V(G) \right\}$$

Proof.

An exercise □

Finding a maximum matching in a graph

Milestone in the history (Edmonds '65)

We can find a maximum matching of a given graph in polynomial time

State of the art

- $\mathrm{O}(\sqrt{n}m \log_n(n^2/m))$ (Goldberg, Karzanov '95)
- $\mathrm{O}(n^{2.376})$ randomized (Mucha, Sankowski '04; Harvey '06)

Open problem

Is it possible to get a *deterministic* $\mathrm{O}(n^{2.376})$ algorithm for this problem, or faster?

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- Open problems

Computing the number of perfect matchings

Fact (Ryser '63)

The number of perfect matchings of a $2n$-vertex bipartite graph can be computed in $\mathrm{O}(2^n \mathrm{poly}(n))$ time

Open Problem

Can we do any faster?

Other facts

- The problem is #P-complete (Valiant '79)
- This value can be efficiently approximated with an arbitrary precision (Jerrum, Sinclair, Vigoda '04: Fulkerson prize winner '06)

Shannon capacity (Shannon '56)

Definition (Strong product)

The strong product $G \times H$ of graphs G, H is defined by

- $V(G \times H) = V(G) \times V(H)$
- $E(G \times H) = \left\{ \{(u,u'),(v,v')\} \;\middle|\; \begin{array}{l} \{u,v\} \in E(G), u'=v' \text{ or} \\ \{u',v'\} \in E(H), u=v \end{array} \right\}$

Definition (Shannon capacity of G)

$$\Theta(G) = \sup_{k \to \infty} (\alpha(\underbrace{G \times G \times \cdots \times G}_{k \text{ times}})^{1/k})$$

Motivated by the determination of an effective size of an alphabet in some noisy communication model

Shannon capacity (continued)

Facts

- $\Theta(C_n) = n/2$ if n even (Not difficult)
- $\Theta(C_5) = \sqrt{5}$ (Lovász '79)

Open Problem

Determine $\Theta(C_7)$

Known: $3.2237 \leq \Theta(C_7) \leq 3.3176$ (see Vesel, Žerovnik '02)

- Determining $\alpha(C_7 \times C_7 \times C_7 \times C_7 \times C_7)$ would improve the lower bound

Shannonc capacity, computationally

Problem (SHANNON CAPACITY)

Input: a graph G and a rational number r
Question: $\Theta(G) \geq r$?

Open Problem (Alon, Lubetzky '06)

Is SHANNON CAPACITY decidable?
(Namely, is there any algorithm to solve SHANNON CAPACITY?)

It's not known the problem is NP-hard