Topics on Computing and Mathematical Sciences I — **April 23, 2008**
Graph Theory Lecture 3: Exercises — **Yoshio Okamoto**

Due Date: April 30, 2008
Legend: $(-)$ easy; $(+)$ hard
Warning: Your solutions have to be substantiated, namely, you have to provide proofs for the given answers.

Exercise 3.1 Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

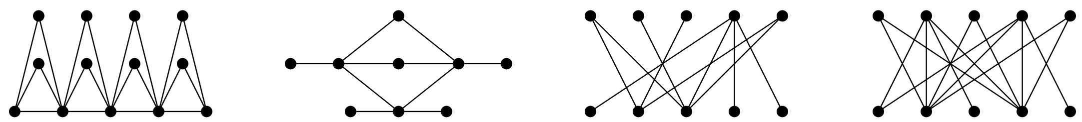
Exercise 3.2 For $n \geq 3$, let G be an n-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine G itself.

Exercise 3.3 (Simulteneous exchangeability of spanning trees) Let T, T' be two spanning trees of a connected graph G. For any $e \in E(T) \setminus E(T')$, prove that there exists an edge $e' \in E(T') \setminus E(T)$ such that both $T' + e - e'$ and $T - e + e'$ become spanning trees of G simultaneously.

Exercise 3.4 Let k be a positive integer.

1. $(+)$ Prove that every n-vertex graph with more than $n(k-1) - \binom{k}{2}$ edges contains all trees with k edges, if $n > k$. (Hint: Try to prove and use the following claim: If $\delta(G) \geq k$ then G contains all trees with k edges.)
2. For every k, construct an n-vertex graph, for some $n > k$, with $n(k-1)/2$ edges which contains no tree with k edges.
3. For $k \in \{1, 2, 3\}$, prove that every n-vertex graph with more than $n(k-1)/2$ edges contains all trees with k edges.

Exercise 3.5 $(-)$ Find a maximum matching in each of the four graphs below. Prove that it is a maximum matching.



Exercise 3.6 Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Exercise 3.7 Prove or disprove: Every tree has at most one perfect matching.