

Matroid Representation of Clique Complexes

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Background

- ◆ An **independence system** can be represented as the intersection of finitely many **matroids**.
- ◆ The **clique complex** of a graph is an independence system.

Question

- ◆ How many matroids do we need for clique complexes?

Motivation

Given later.

V a nonempty finite set

Def.: an **independence system** on V is
a nonempty family \mathcal{I} of subsets of V s.t.

$$X \in \mathcal{I} \implies Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Example: $V = \{1, 2, 3, 4\};$

Which is an independence system?

$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}\}$
 $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}\}.$

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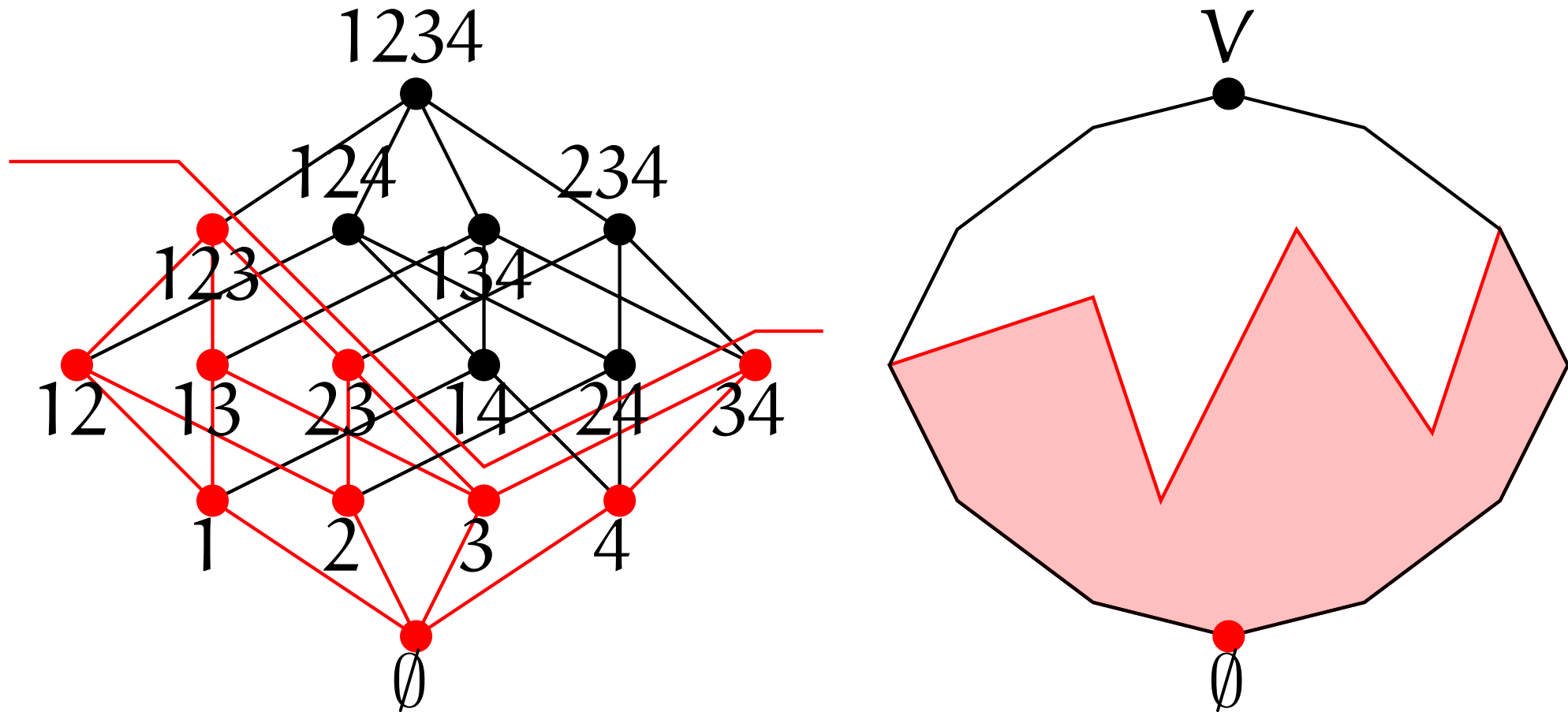
$$X \in \mathcal{I} \implies Y \in \mathcal{I} \quad \text{for all } Y \subseteq X.$$

Remark:

also called an abstract simplicial complex.

- ◆ From graphs
 - the family of the cliques
 - the family of the forests
 - the family of the matchings
- ◆ From partially ordered sets
 - the family of all chains
- ◆ From polytopes
 - the family of all faces of a simplicial polytope
- ◆ From game theory
 - the family of losing coalitions of a weighted majority game

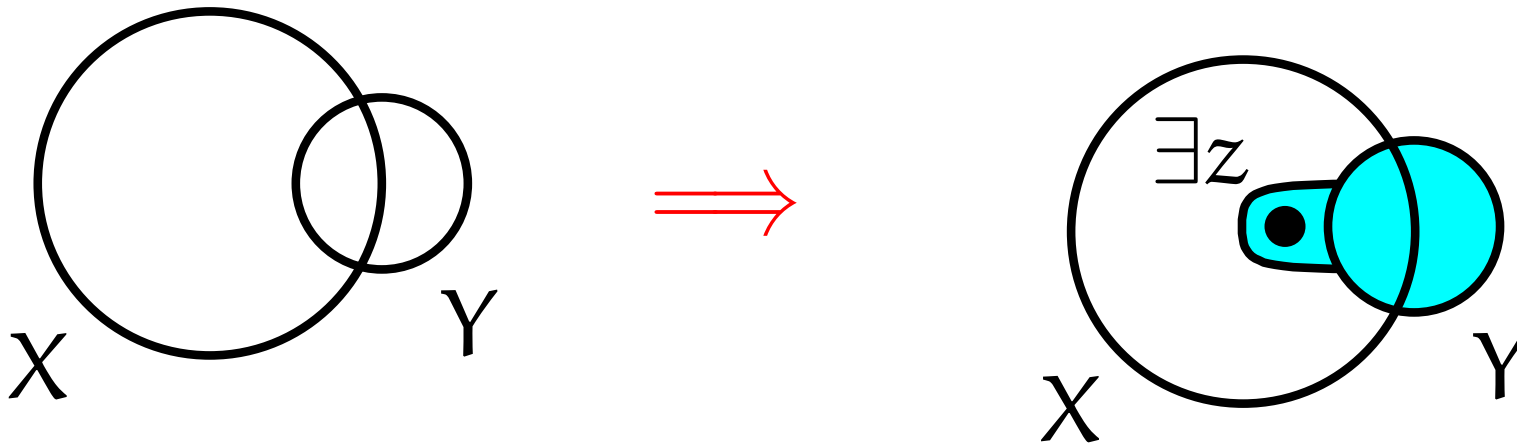
$$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 34, 123\}.$$



(known as the **face poset** of \mathcal{I} ,
when we see \mathcal{I} as an abstract simplicial complex)

Def.: An indep. system \mathcal{I} is called a **matroid** if \mathcal{I} satisfies the **augmentation axiom**:

$$X, Y \in \mathcal{I}, |X| > |Y| \implies \exists z \in X \setminus Y \text{ s.t. } \underline{Y \cup \{z\}} \in \mathcal{I}.$$

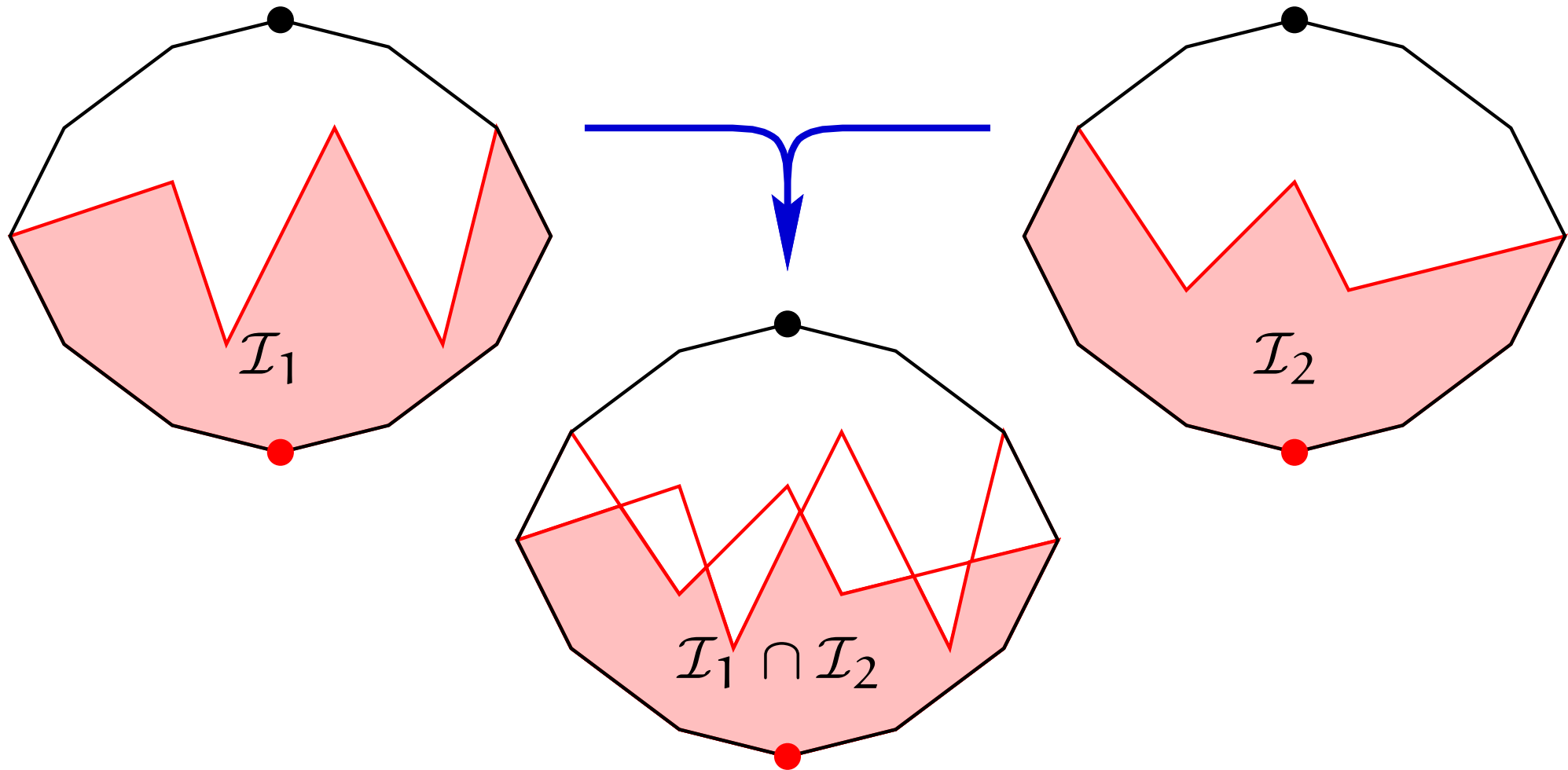


Appearance of matroids:

Combinatorial optimization, Combinatorial geometry,
Combinatorial topology, Coding theory, Design theory, etc.

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \{X \subseteq V : X \in \mathcal{I}_1 \text{ and } X \in \mathcal{I}_2\}.$$





Intersection of independence systems

The intersection of indep. systems $\mathcal{I}_1, \mathcal{I}_2$ on V is

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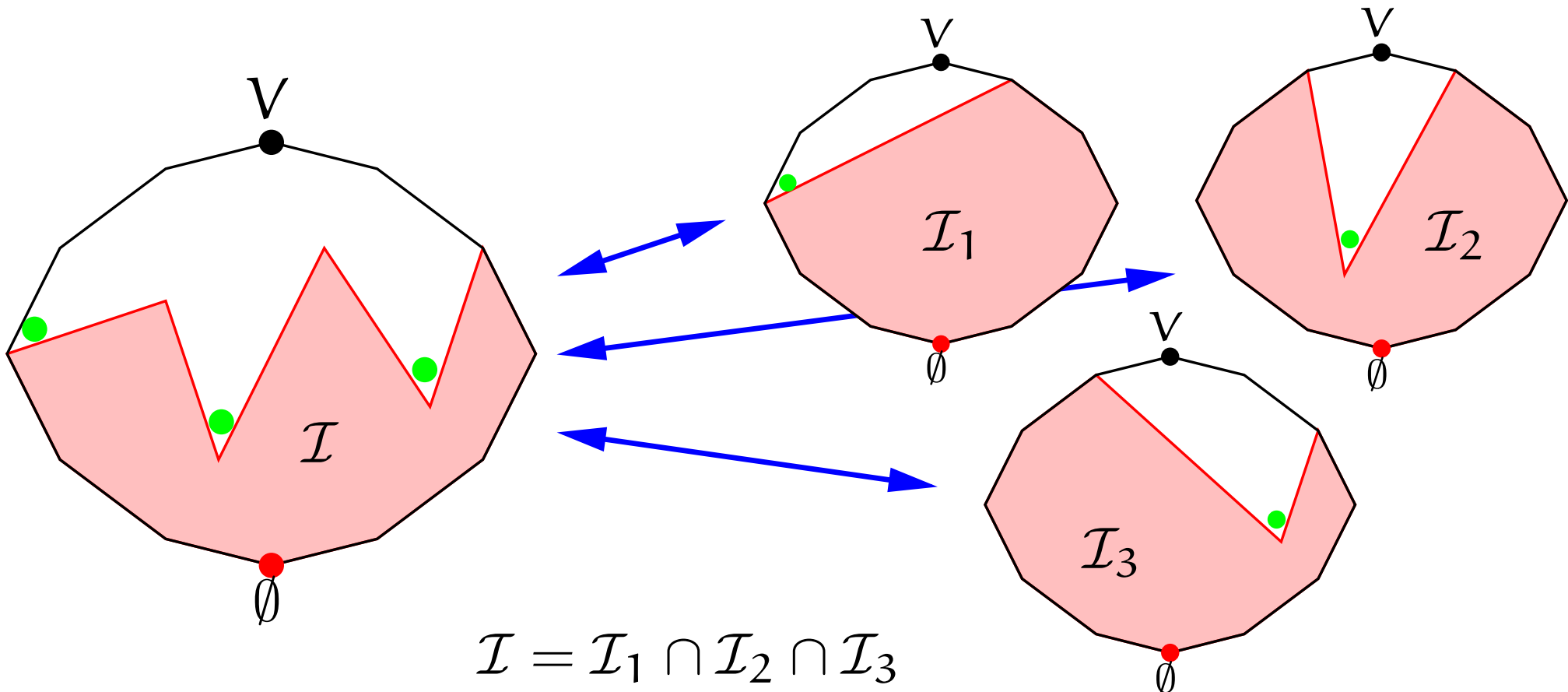
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- ◆ The intersection of more indep. systems is defined analogously.
- ◆ The intersection of indep. systems is again an indep. system.
- ◆ The intersection of matroids is an indep. system but not necessarily a matroid. However ...

Fact: Every indep. system is the intersection of finitely many matroids.

Proof. Min. sets not in \mathcal{I} are shown by \bullet .





Question:

How many matroids do we need to represent a given indep. system as their intersection??

.....

The proof in the prev. page says that:

this number \leq the number of \bullet for \mathcal{I} .

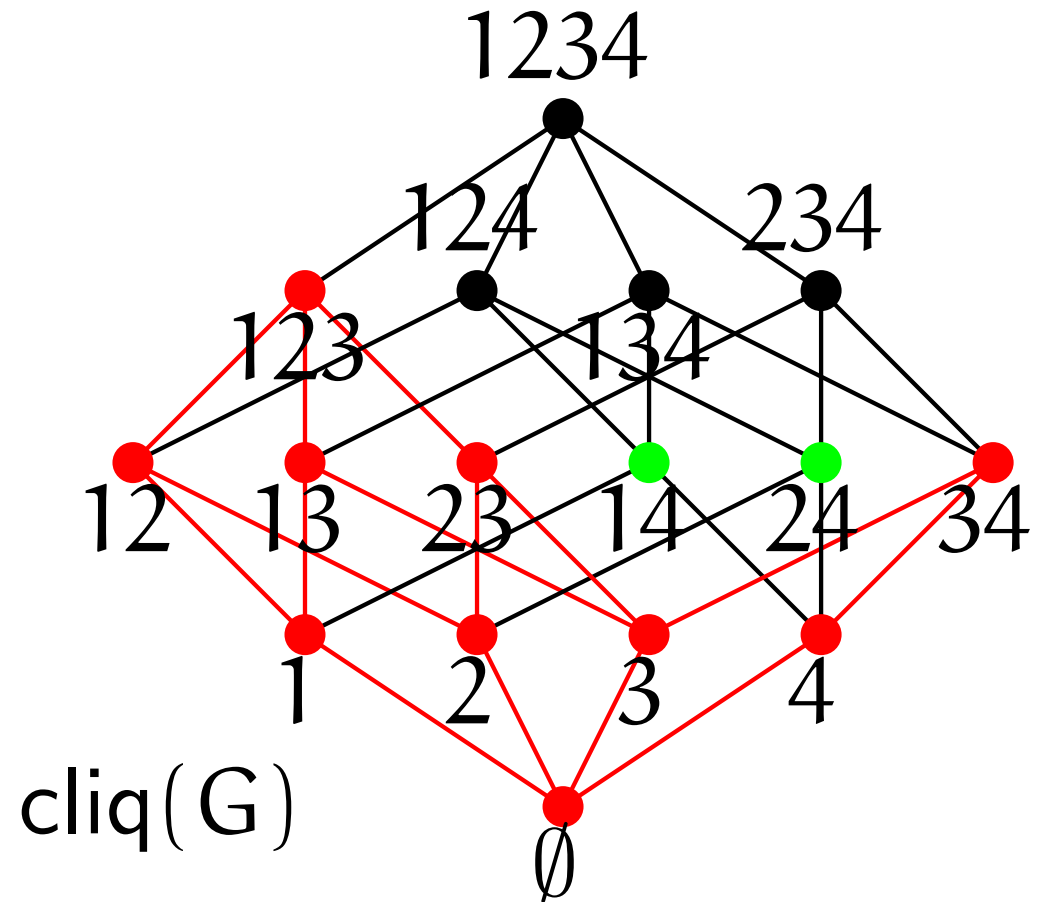
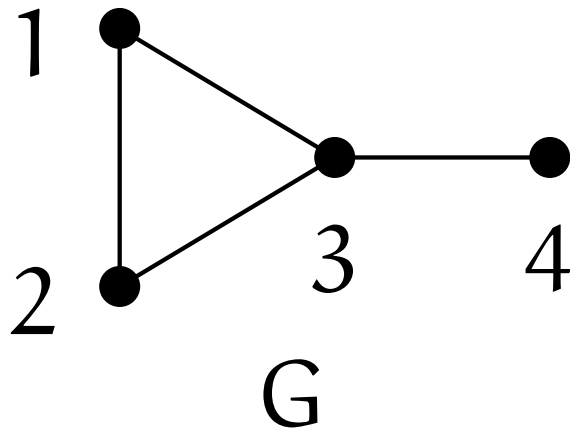
But, we might do better.

★ We study this number for **clique complexes**.

Def.: The **clique complex** of a graph $G = (V, E)$ is the family of all cliques of G .

Example:

$$V = \{1, 2, 3, 4\}$$



Question:

How many matroids do we need to represent a clique complex as their intersection??

Result:

We investigate and characterize these numbers.

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However, you may ask...

- (1) Why do we consider such a representation??
- (2) Why do we consider clique complexes??

Q. Why do we consider such a representation?

Ans. in the following proposition.

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Prop.: (Jenkyns, '76; Korte–Hausmann, '78)

\mathcal{I} an independence system

\mathcal{I} is the intersection of k matroids

\implies Greedy Alg. is a k -approx. alg. for
Max weighted base problem for \mathcal{I} .

Q. Why do we consider such a representation?

Ans. in the following proposition.

Prop.: (Jenkyns, '76; Korte–Hausmann, '78)

$\text{cliq}(G)$ the clique complex of G

$\text{cliq}(G)$ is the intersection of k matroids

\implies Greedy Alg. is a k -approx. alg. for
Max weighted clique problem in G .

★ $k \approx$ how complex G is w.r.t. clique problem.

Q. Why do we consider clique complexes?

Q. Why do we consider clique complexes?

- Ans. The class of clique complexes includes some important classes like the classes of
- ◆ the family of the matchings of a graph G
= $\text{cliq}(\text{the complement of the line graph of } G)$
studied by Fekete–Firla–Spille '03
in the same framework as ours.
 - ◆ the family of the chains of a poset P
= $\text{cliq}(\text{the comparability graph of } P)$
 - ◆ ...

In the rest of my talk

- (1) clique complex
which is a matroid
 - ◆ Key concept: partition matroid
- (2) clique complex
which is the intersection of k matroids
 - ◆ Key concept: stable-set partition
- (3) other results

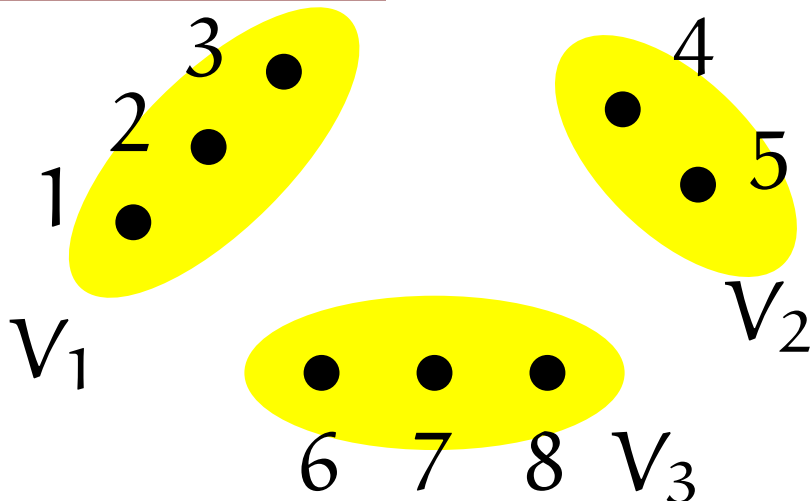
V a nonempty finite set

$\mathcal{P} = \{V_1, V_2, \dots, V_r\}$ a partition of V

Def.: A **partition matroid** $\mathcal{I}(\mathcal{P})$ of \mathcal{P} is

$\mathcal{I}(\mathcal{P}) = \{X \subseteq V : |X \cap V_i| \leq 1 \text{ for all } i = 1, \dots, r\}.$

Example: $\mathcal{P} = \{V_1, V_2, V_3\}$



$\mathcal{I}(\mathcal{P}) =$
 $\{146, 147, 148, 156, 157, 158,$
 $246, 247, 248, 256, 257, 258,$
 $346, 347, 348, 356, 357, 358,$
 and their subsets}

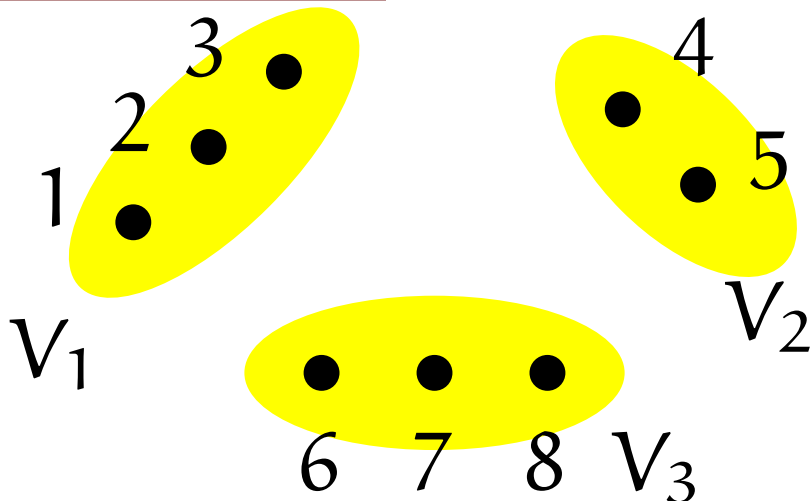
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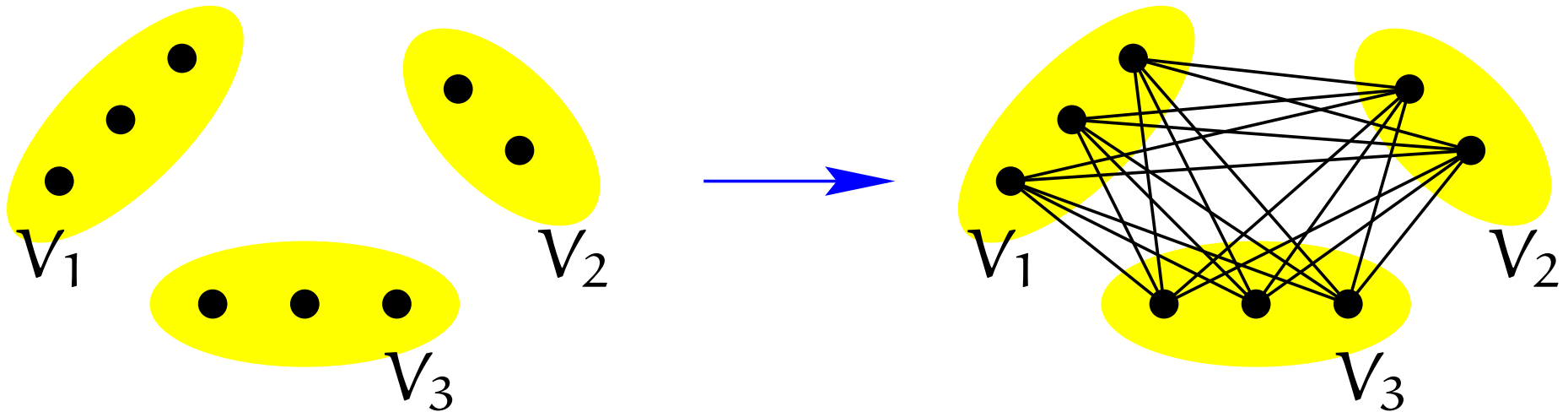
Example: $\mathcal{P} = \{V_1, V_2, V_3\}$



Notice: In fact,
 $\mathcal{I}(\mathcal{P})$ is a matroid.

Lem.: Partition matroids are clique complexes.

Proof. Given a partition $\mathcal{P} = \{V_1, \dots, V_r\}$ of V .



Construct a complete r -partite graph $G_{\mathcal{P}}$ s.t. each part of $G_{\mathcal{P}}$ corresponds to V_i .

Then, $\text{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$.

[qed]

Remark:

G a graph

G is complete multipartite



$\text{cliq}(G)$ is a matroid.

Thm 1:

G a graph

G is complete multipartite



$\text{cliq}(G)$ is a matroid.

Proof.

Omitted.

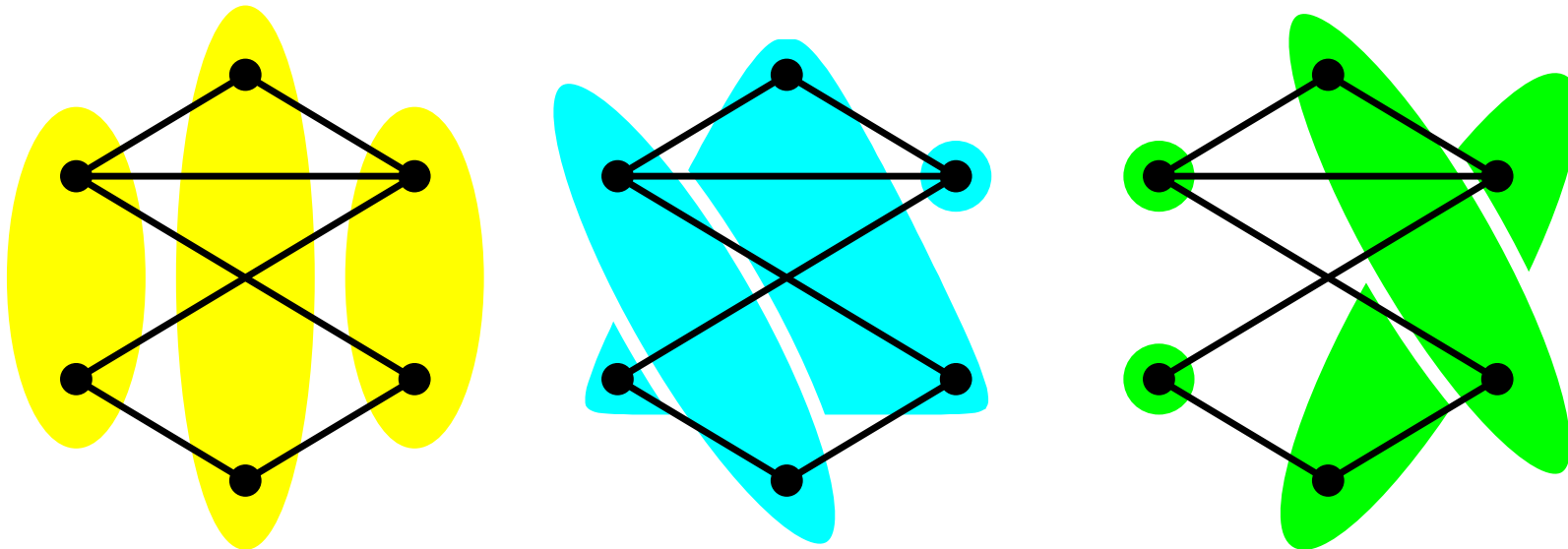
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- (2) clique complex
which is the intersection of k matroids
 - ◆ Key concept: stable-set partition
- (3) other results

$G = (V, E)$ a graph

Def.: A **stable-set partition** of G
is a partition $\mathcal{P} = \{V_1, \dots, V_r\}$ of V
s.t. V_i is a stable set of G ($i = 1, \dots, r$).

Example:



²¹√ Construct a graph from a stable-set partition

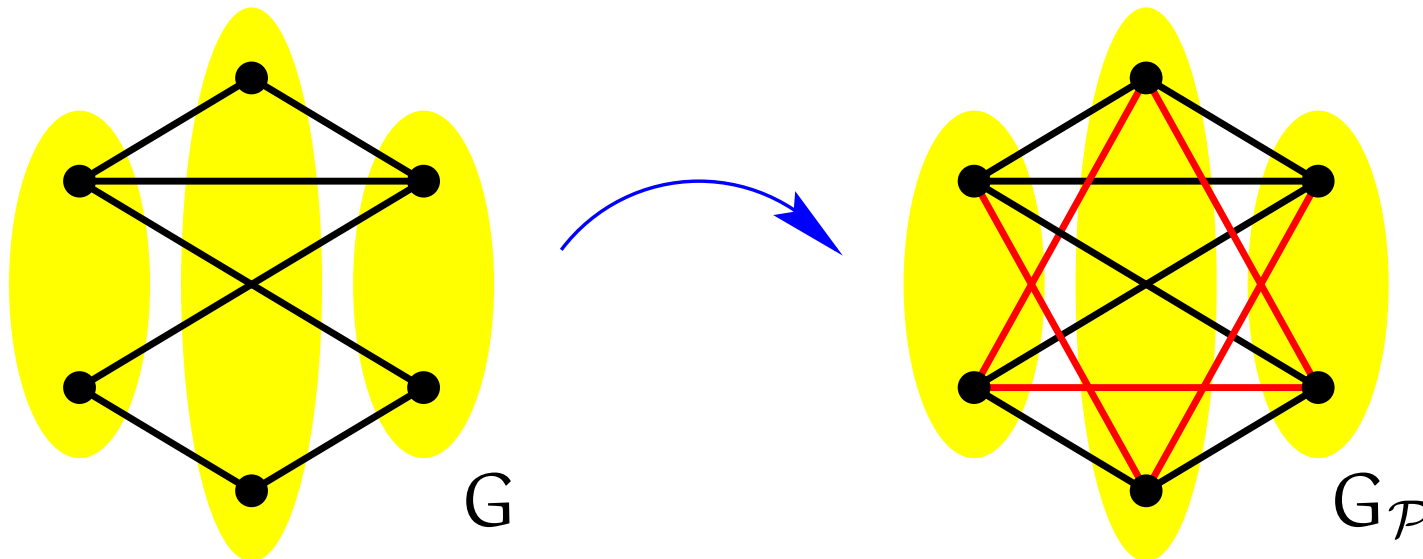
G a graph

\mathcal{P} a stable-set partition of G

Construct a complete r -partite graph $G_{\mathcal{P}}$ from \mathcal{P} .

Observation: An edge of G is an edge of $G_{\mathcal{P}}$.

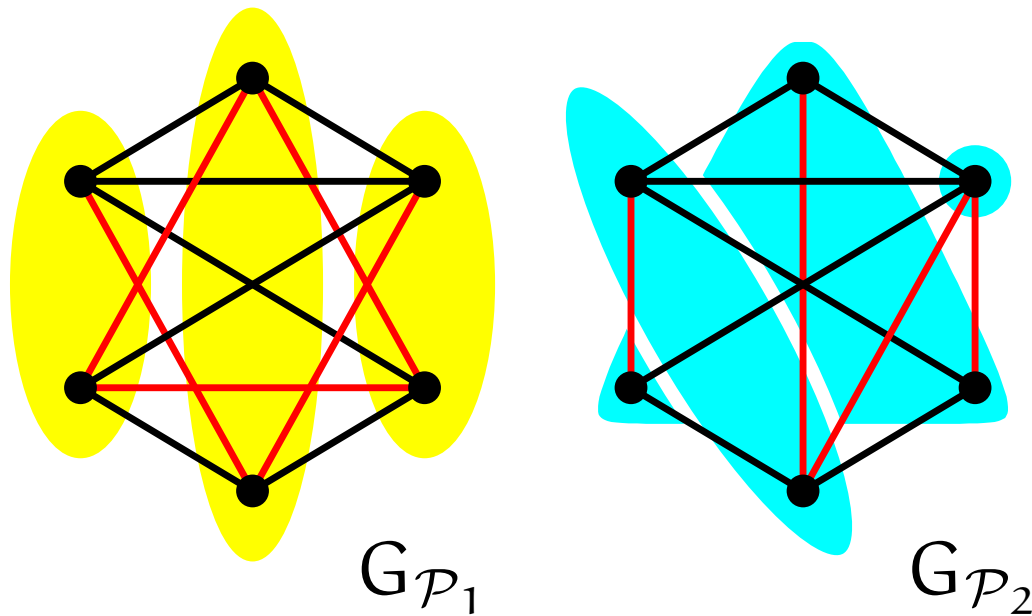
This means: $\text{cliq}(G) \subseteq \text{cliq}(G_{\mathcal{P}}) = \mathcal{I}(\mathcal{P})$.



Generally,

$$\text{cliq}(G) \subseteq \text{cliq}(G_{\mathcal{P}_1}) \cap \text{cliq}(G_{\mathcal{P}_2}).$$

In this example,

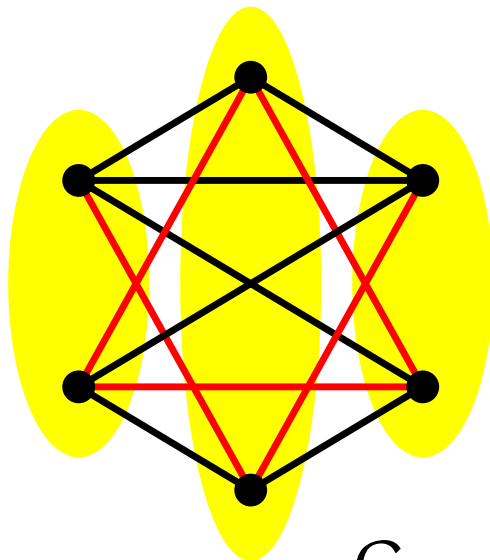
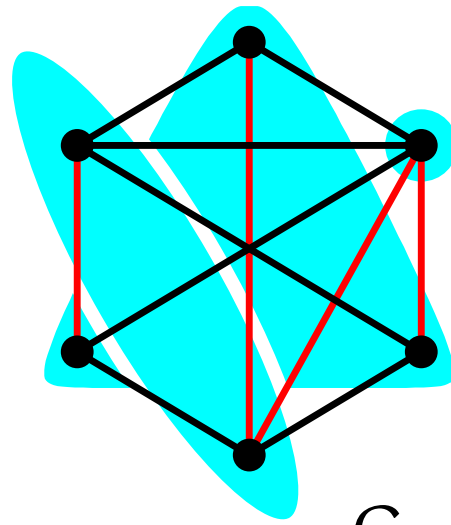
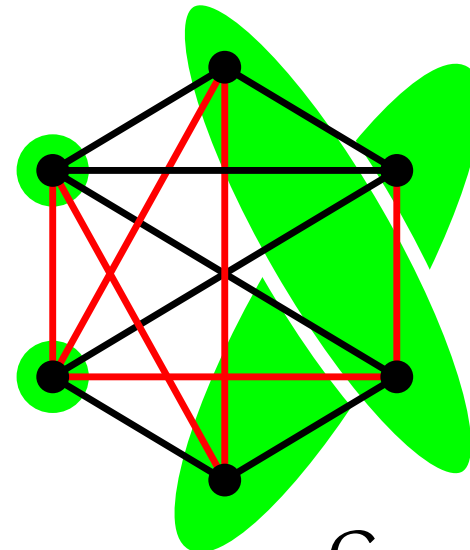


$$\text{cliq}(G) \neq \text{cliq}(G_{\mathcal{P}_1}) \cap \text{cliq}(G_{\mathcal{P}_2})$$

Generally,

$$\text{cliq}(G) \subseteq \text{cliq}(G_{\mathcal{P}_1}) \cap \text{cliq}(G_{\mathcal{P}_2}) \cap \text{cliq}(G_{\mathcal{P}_3}).$$

However, in this example,


 $G_{\mathcal{P}_1}$

 $G_{\mathcal{P}_2}$

 $G_{\mathcal{P}_3}$

$$\text{cliq}(G) = \text{cliq}(G_{\mathcal{P}_1}) \cap \text{cliq}(G_{\mathcal{P}_2}) \cap \text{cliq}(G_{\mathcal{P}_3})$$

\therefore No red edge is contained in all of these graphs.

$G = (V, E)$ a graph

$\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ stable-set partitions of G

$$\text{cliq}(G) = \bigcap \{I(\mathcal{P}_i) : i = 1, \dots, k\}$$



No red edge is contained in all $G_{\mathcal{P}_i}$'s.

Remark: $G = (V, E)$ a graph

\exists k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ of G s.t.
 $\text{cliq}(G) = \bigcap \{\mathcal{I}(\mathcal{P}_i) : i = 1, \dots, k\}$



\exists k stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ of G s.t.
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Thm 2: $G = (V, E)$ a graph

$\text{cliq}(G)$ is the intersection of k matroids



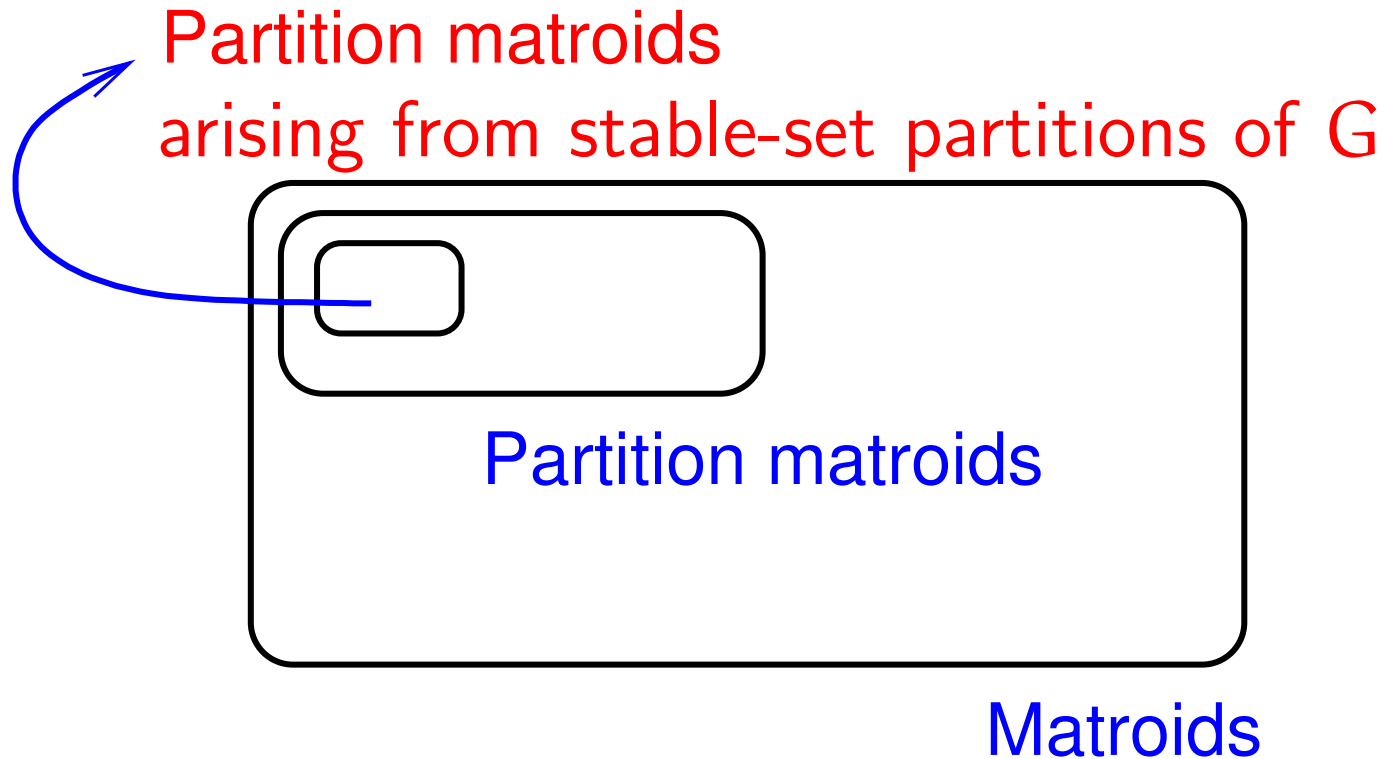
$\exists k$ stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ of G s.t.
 $\text{cliq}(G) = \bigcap \{\mathcal{I}(\mathcal{P}_i) : i = 1, \dots, k\}$



$\exists k$ stable-set partitions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ of G s.t.
no **red edge** is contained in all $G_{\mathcal{P}_i}$'s.

Proof. Omitted.

Given a graph G .



When we want a representation of $\text{cliq}(G)$ by k matroids, just search k partition matroids from stable-set partitions of G .

Consider the following decision problem.

Instance: a graph G and a positive integer k

Question: ?? $\exists k$ matroids $\mathcal{I}_1, \dots, \mathcal{I}_k$
s.t. $\text{cliq}(G) = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_k$??

Thm 2 implies: **This problem $\in \mathcal{NP}$.**

In the rest of my talk

- (1) clique complex
which is a matroid
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(five more pages!!!)

Thm 3: G a graph

$\text{cliq}(G)$ is the intersection of two matroids



The stable-set graph of G is bipartite.

Thm 3: G a graph

$\text{cliq}(G)$ is the intersection of two matroids



The stable-set graph of G is bipartite.

Note: Combining $\left\{ \begin{array}{l} \text{Theorem 3 and} \\ \text{Protti–Szwarcfiter ('02),} \end{array} \right.$

we can tell, in polynomial time, whether

the clique complex of a given graph
is the intersection of two matroids or not.

Consider the following decision problem.

Fix: a positive integer k

Instance: a graph G

Question: ? $\exists k$ matroids $\mathcal{I}_1, \dots, \mathcal{I}_k$

s.t. $\text{cliq}(G) = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_k$?

What is the complexity of this problem for each k ?

k	1	2	3	4	...
	\mathcal{P}	\mathcal{P}	?	?	...

A graph can be seen as an independence system:

$$X \in G \iff X = \begin{cases} \emptyset & \text{or} \\ \{v\} & \text{or} \\ \{u, v\} & \text{if it's an edge.} \end{cases}$$

Thm 4: G a graph

G is the intersection of k matroids



$\text{cliq}(G)$ is the intersection of k matroids.

Let $k(G) = \min\{k : \text{cliq}(G) \text{ is the intersection of } k \text{ matroids}\}.$

$$k(n) = \max\{k(G) : G \text{ has } n \text{ vertices}\}.$$

Thm 5: $k(n) = n - 1.$

Question

- ◆ How many matroids do we need to represent a given clique complex as their intersection??

Result

- ◆ The characterization of graphs whose clique complexes are the intersections of k matroids

[End of the talk]

Tak!

