Fair cost allocations under conflicts
— a game-theoretic point of view —

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Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  . . . . optimization problem
- They want to allocate the benefit in a fair way.
  . . . . game-theoretic problem

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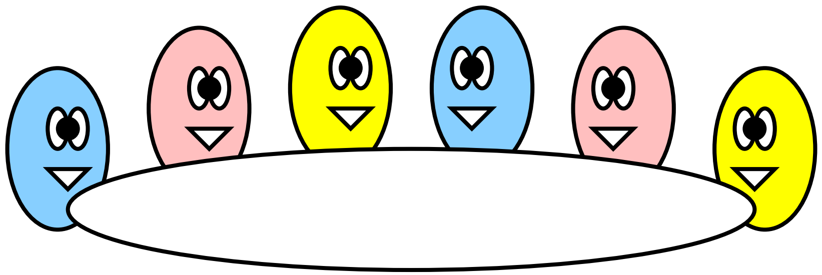
Game Theory?

- Noncooperative Game Theory
- Cooperative Game Theory



Def.: A **cooperative game** (or a **game**) is a pair (N, γ) of

- a finite set N (set of players)
- a function $\gamma : 2^N \to \mathbb{R}$ with $\gamma(\emptyset) = 0$ (characteristic function).

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Interpretation: For $S \subseteq N$,

$\gamma(S)$ represents $\left\{ \begin{array}{l} \text{the max. benefit gained by } S \\ \text{the min. cost owed by } S \end{array} \right\}$
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Goal: To allocate $\gamma(N)$ to each player in a "fair" way.

This work: study on "minimum coloring games."

$G = (V, E)$ an undirected graph

- A **proper k-coloring** of G

  is a surjective map $c : V \to \{1, \ldots, k\}$ s.t.
  if $\{u, v\} \in E$, then $c(u) \neq c(v)$.

- The **chromatic number** $\chi(G)$ of G

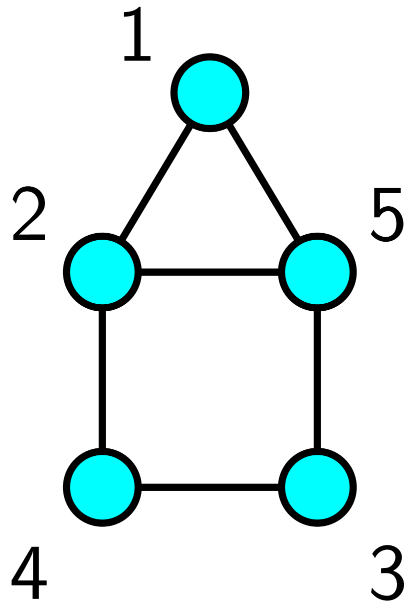
  $= \min\{\ k\ :\ $ a proper k-coloring of G exists $\}$.

- The **minimum coloring game** on G

  is a cooperative game (V, χ_G).

  $\chi_G : 2^V \to \mathbb{N}$ is defined as $\chi_G(S) = \chi(G[S])$,
  where $G[S]$ is the subgraph induced by $S \subseteq V$.

$\chi_G(S) = \chi(G[S])$ for $S \subseteq V$.

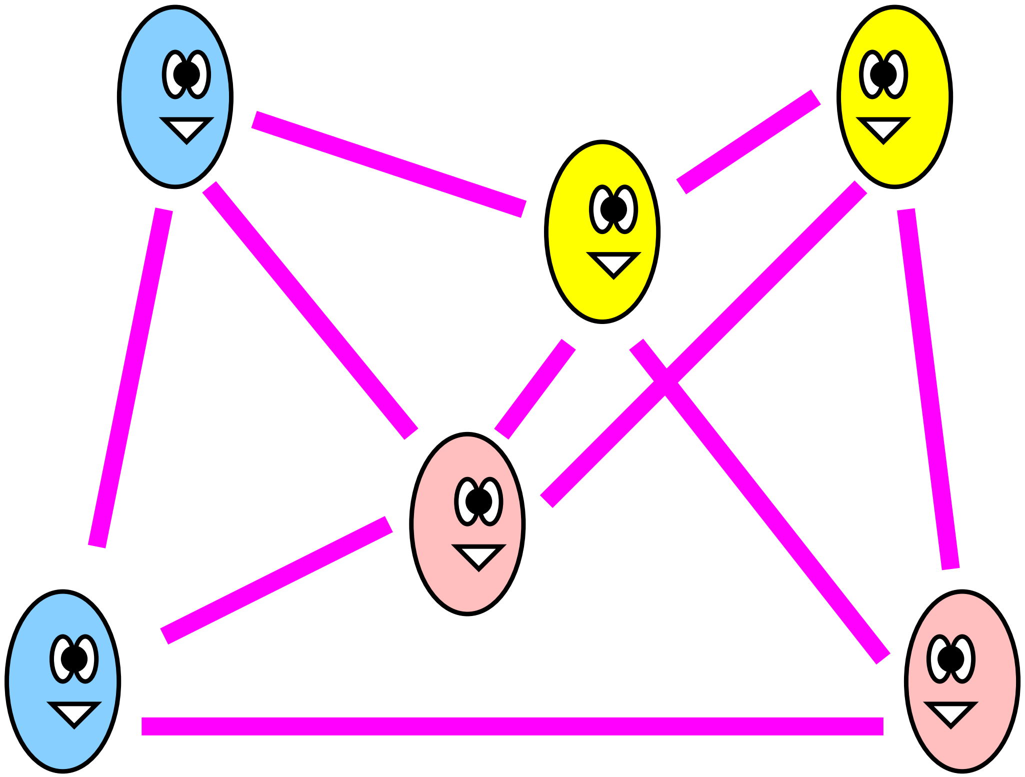


S	χ_G	S	χ_G	S	χ_G	S	χ_G
$\emptyset$	0	14	1	123	2	245	2
1	1	15	2	124	2	345	2
2	1	23	2	125	3	1234	2
3	1	24	1	134	2	1235	3
4	1	25	2	135	2	1245	3
5	1	34	2	145	2	1345	2
12	2	35	1	234	2	2345	2
13	1	45	2	235	2	12345	3

Goal: To allocate $\chi(G)$ to each vertex in a fair way.

Conflict graph: a model of conflict

- the vertices = the agents, the principals...
- the edges = between two in conflict.



min. coloring game:

a simplest model of the
fair cost allocation problem
in conflict situations

We study minimum coloring games, and
investigate the following kinds of fairness concepts:

- Core (Gillies '53)
- Nucleolus (Schmeidler '69)
- τ-value (Tijs '81)
- Shapley value (Shapley '53).

Past works on minimum coloring games:

- Deng, Ibaraki & Nagamochi '99
- Deng, Ibaraki, Nagamochi & Zang '00
- Okamoto '03.

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Why fair cost allocation problems??

Fair cost allocation problems are studied in OR community from the game-theoretic viewpoint.

- Megiddo '87

  First noticed the computational issue
  in fair cost allocation problems.

- So far, a lot of results have appeared in

  Mathematics of Operations Research,
  Mathematical Programming,
  Mathematical Methods of Operations Research,
  Discrete Applied Mathematics,
  International Journal of Game Theory,
  Games and Economic Behavior,
  etc.

- They assume practical applications.

There are many kinds of "fairness" concepts
(called "solutions") in cooperative game theory.

Q. Which fairness concept should be used better??

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Thesis: Bounded Rationality (Simon '70s)

Decisions by realistic economic agents cannot involve unbounded resources for reasoning.

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Thesis: (Deng & Papadimitriou '94)

[Economic concept]		[Algorithmic concept]
A solution follows Bounded Rationality	$\simeq$	Computation can be done in poly. time

$\Longrightarrow$ Algorithmic study of cooperative game theory

Combinatorial Optimization Games

= cooperative games arising from combinatorial optimization problems

- Assignment games (Shapley & Shubik '71)
- Minimum-cost spanning tree games (Bird '76)
- Steiner tree games (Megiddo '78)
- Simple flow games (Kalai & Zemel '82)
- Travaling salesman games (Potters, Curiel & Tijs '92)
- Chinese postman games (Granot, Hamers & Tijs '99)
- Facility location games (Goemans & Skutella '00)
- Minimum coloring games (Deng, Nagamochi & Ibaraki '99)
- ......

E a finite set

Def.: A **set function** on E is a function $f : 2^E \to \mathbb{R}$.

Appearance:

- Cooperative game theory
- Combinatorial optimization
- Pseudo-boolean functions
- Nonadditive measure theory (fuzzy measure theory)
- ...

They study different aspects of set functions.

Focus on cores and nucleoli of minimum coloring games

- Def.: cost allocation
- Def.: core
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Def.: A **cost allocation** for a game (N, γ) is a vector $z \in \mathbb{R}^N$ such that

$$\sum\{z[i] : i \in N\} = \gamma(N).$$

Interpretation:

$z[i] =$ the amount of cost the player i must pay when all players in N work together

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. . . .

Q. What kinds of cost allocations are considered fair??

…… Core, Nucleolus, τ-value, Shapley value, etc.

Def.: A cost allocation $z \in \mathbb{R}^N$ for (N, γ) is a **core allocation** if

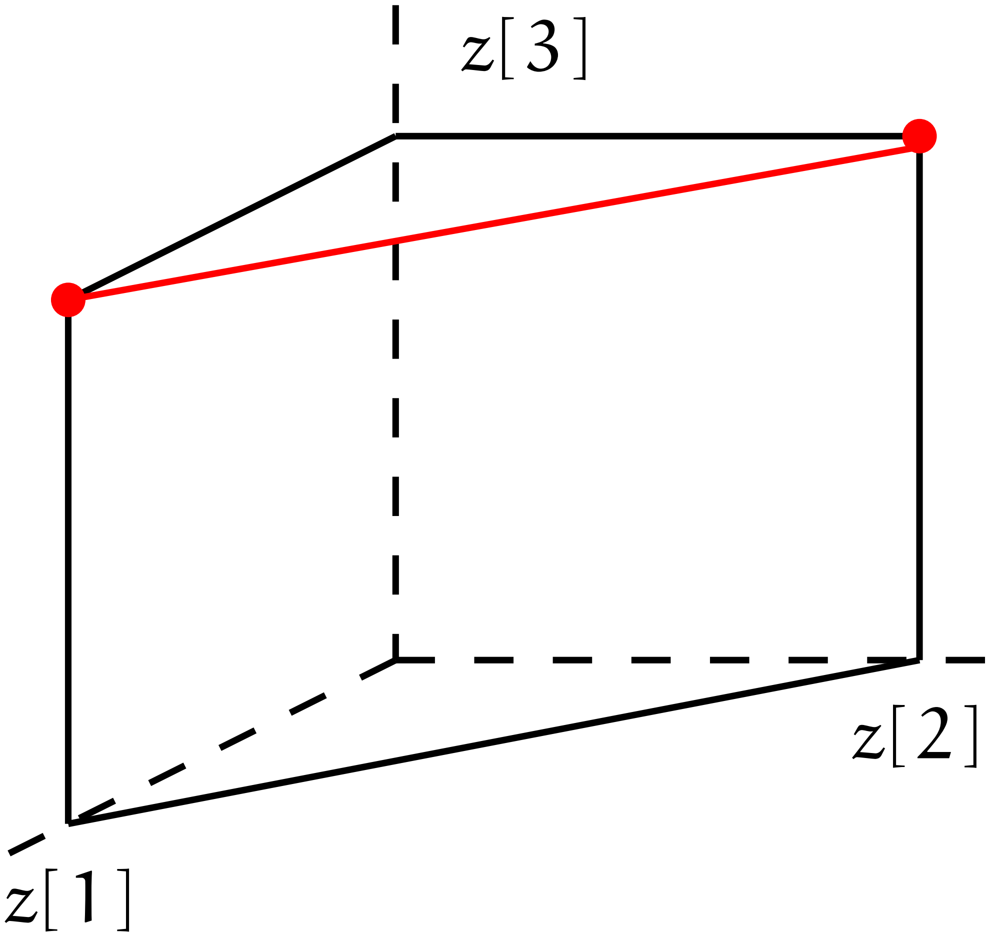
$$\sum \{z[i] : i \in S\} \leq \gamma(S) \quad \text{for all } S \subseteq N$$

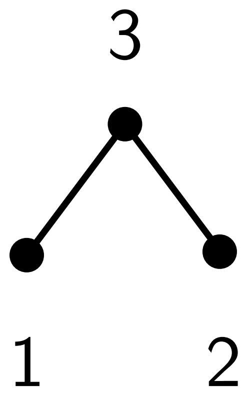
The **core** of (N, γ) is the set of all core allocations.

Interpretation: Each subset $S \subseteq N$ is satisfied with z

$\sum_{i \in S} z[i]$: cost owed by S when people in N work together

$\gamma(S)$: cost owed by S when people in S work together.

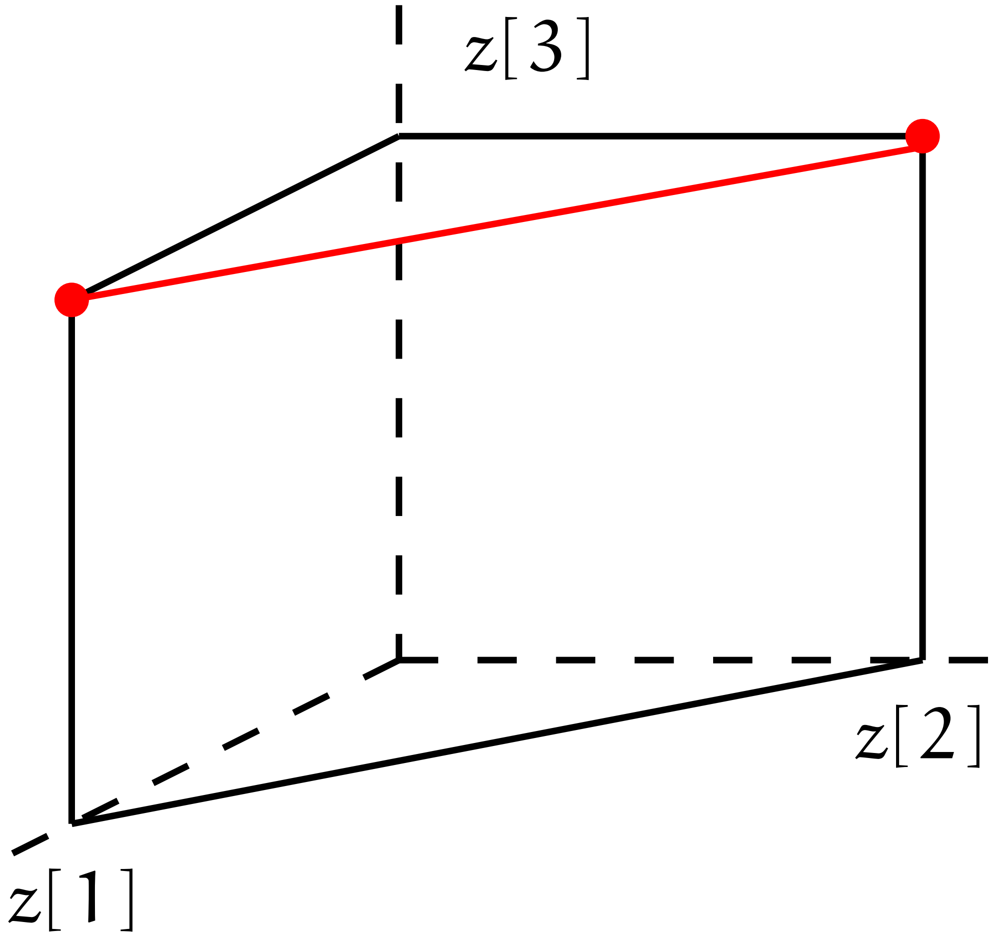


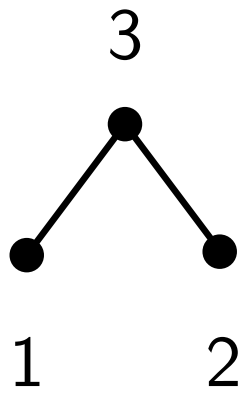


$V = \{1, 2, 3\}$

$\chi_G(\emptyset)$	0
$\chi_G(\{1\})$	1
$\chi_G(\{2\})$	1
$\chi_G(\{3\})$	1
$\chi_G(\{1,2\})$	1
$\chi_G(\{1,3\})$	2
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$$\text{Core} = \left\{ z \in \mathbb{R}^3 : \begin{array}{l} z[1] \leq 1,\ z[2] \leq 1,\ z[3] \leq 1, \\ z[1] + z[2] \leq 1,\ z[1] + z[3] \leq 2, \\ z[2] + z[3] \leq 2,\ z[1] + z[2] + z[3] = 2 \end{array} \right\}$$





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$\chi_G(\{1,3\})$	2
$\chi_G(\{2,3\})$	2
$\chi_G(\{1,2,3\})$	2

$$\text{Core} = \text{conv}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

We want to know:

(1) When is the core empty??

(2) If the core is nonempty,

  (a) What are the extreme points of the core??
  (b) How can we compute a core allocation??

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(b) How can we compute a core allocation??

(1) Thm. (Deng, Ibaraki & Nagamochi '99)

It is NP-complete to decide the min coloring game of a given graph has a nonempty core or not.

(2) Thm. (Deng, Ibaraki & Nagamochi '99)

For a bipartite graph G,
core = conv(the char. vectors of edges of G).

We will generalize it to ...

Def.: G is **perfect** if for every induced subgraph $H \subseteq G$
$\chi(H) =$ max size of the cliques in H.

(A clique is a vertex subset which induces a complete graph.)

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Thm. (Deng, Ibaraki, Nagamochi & Zang '00)

A min coloring game (V, χ_G) is totally balanced

$\Updownarrow$

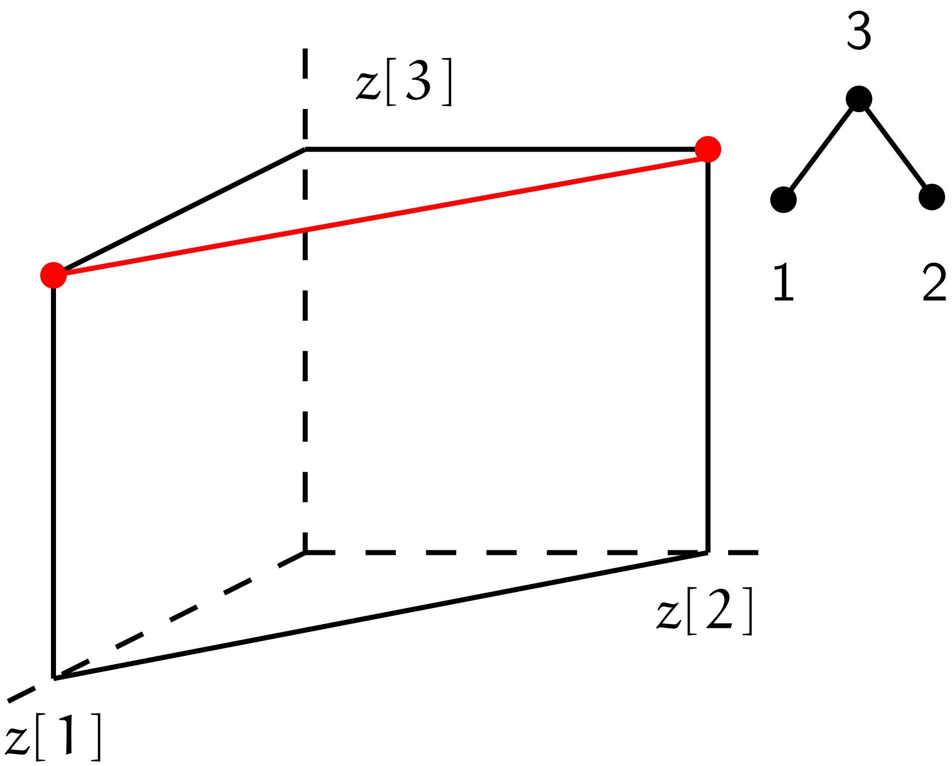
G is perfect.

Rem.: A game is totally balanced
$\Longrightarrow$ it has a nonempty core.

Thm.

G a perfect graph

core $=$ conv(the char. vectors of the maximum cliques of G).



$$\text{Core} = \text{conv}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Cor. We can do the following in poly. time.

(1) Find a core allocation for a perfect graph.
($\approx$ Find a maximum clique in poly. time.)

(2) Decide whether a given vector belongs to the core or not for a perfect graph.
($\approx$ The membership problem for the clique polytope.)

They are the consequences of the previous theorem and a result by Grötschel, Lovász & Schrijver ('83).

What is good/bad for the core??

Good :)

- Easy to investigate.
- Much is known.

Bad :(

- Might be empty.
- Even if not empty,
  there might be many allocations in the core.
  (We need another criterion to choose one of them.)

The nucleolus is another fairness concept, which uniquely exists for every min coloring game.

Focus on cores and nucleoli of minimum coloring games

- Def.: cost allocation
- Def.: core

. . . .

- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Let (N, γ) a game
$z \in \mathbb{R}^N$ a cost allocation
$S \subseteq N$ (often called a coalition)

Def.: An **excess** $e(S, z)$ is defined as

$$e(S, z) := \sum_{i \in S} z[i] - \gamma(S).$$

Interpretation:

(1) $z \in$ core $\iff e(S, z) \le 0$.
(2) The smaller $e(S, z)$, the happier S with z.
(3) $e(S, z) \sim$ how much z satisfies the condition of the core for S.

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.

Example:

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

S	$\gamma(S)$	$e(S, z)$
$\emptyset$	0	(0)
$\{1\}$	1	0
$\{2\}$	1	$-1/2$
$\{3\}$	1	$-1/2$
$\{1, 2\}$	1	$1/2$
$\{1, 3\}$	2	$-1/2$
$\{2, 3\}$	2	-1
$\{1, 2, 3\}$	2	(0)

Let (N, γ) be a game, $z \in \mathbb{R}^N$ a cost allocation

Consider the following procedure.

- Enumerate $e(S, z)$ for all $S \in 2^N \setminus \{\emptyset, N\}$.
- Arrange these excesses in non-increasing order to obtain $\theta_z \in \mathbb{R}^{2^{|N|}-2}$. ($\theta_z[i] \geq \theta_z[j]$ if $i \leq j$.)

Example:

$$z = \left(1, \frac{1}{2}, \frac{1}{2}\right)^\top$$

$$\theta_z = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -1\right)^\top$$

S	$\gamma(S)$	$e(S, z)$
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$\{2,3\}$	2	-1
$\{1,2,3\}$	2	(0)

Def.: The **nucleolus** of (N, γ) is defined as

$$\boldsymbol{\nu}(N,\gamma) = \left\{ \boldsymbol{z} \in \mathbb{R}^N : \begin{array}{l} \boldsymbol{z} \text{ lex-mins } \theta_{\boldsymbol{z}} \text{ over all cost alloc's } \boldsymbol{y} \\ \text{s.t. } \boldsymbol{y}[i] \le \gamma(\{i\}) \quad \forall i \in N \end{array} \right\}.$$

Interpretation: The smaller $e(S, \boldsymbol{z})$, the happier S with $\boldsymbol{z}$.

$\Rightarrow$ Want an allocation which minimizes max excess.

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Interpretation: The smaller $e(S, z)$, the happier S with z.

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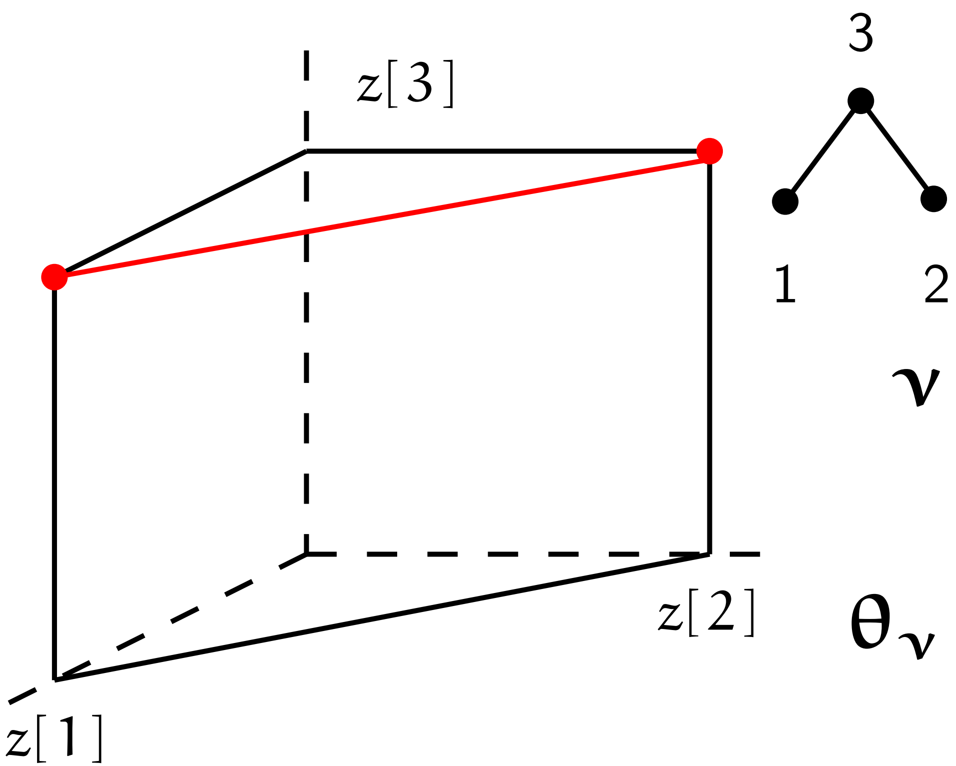
Thm. (Schmeidler '69)

The nucleolus consists of a single vector.

So we usually say $\nu(N, \gamma) = z$ instead of $\nu(N, \gamma) = \{z\}$.

Example: nucleolus



$$\nu = \left(\frac{1}{2}, \frac{1}{2}, 1\right)^\top$$ is the nucleolus.

$$\theta_\nu = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^\top.$$

Fact: the core is nonempty

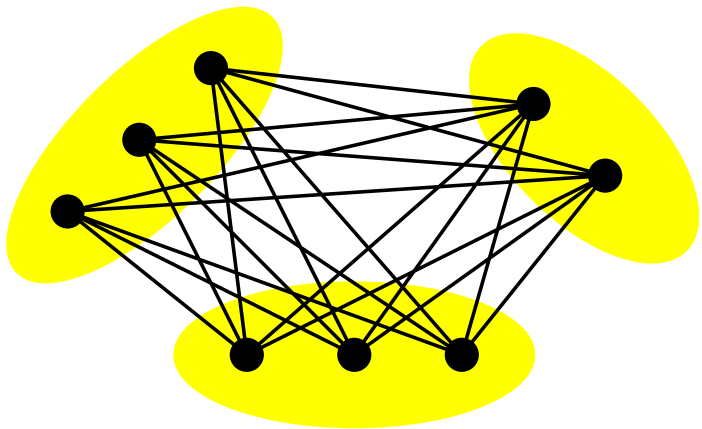
$\Rightarrow$ the nucleolus $\in$ the core.

Thm. (Kuipers '96, Faigle, Kern & Kuipers '01)

The nucleolus can be computed in polynomial time for submodular games.

Thm. (Okamoto '03)

χ_G is submodular $\Leftrightarrow$ G is complete multipartite.



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Cor.

G complete multipartite
$\Rightarrow$ the nucleolus of χ_G computed in poly. time.

On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\!\!\mid$		
perfect		???
$\cup\!\!\mid$		
complete multipartite		Poly

On the computation of the nucleolus of a min coloring game

Graph	$\leftrightarrow$	Min col. game
general		NP-hard
$\cup\!\mid$		
perfect		???
$\cup\!\mid$		
O-good		characterization
$\cup\!\mid$		
complete multipartite		Poly

Thm.

The nucleolus for an O-good perfect graph G is the barycenter of the characteristic vectors of the maximum cliques of G.

Namely,

$$\boldsymbol{\nu}[i] = \frac{\#\text{ of maximum cliques containing } i}{\#\text{ of maximum cliques}}.$$

Thm.

The nucleolus for an O-good perfect graph G
is the barycenter of the characteristic vectors of
the maximum cliques of G.

Remarks:

(1) We omit the def. of O-good perfect graphs.

(2) The class of O-good perfect graphs contains

- the graphs with unique maximum cliques
- the complete multipartite graphs
- the chordal graphs (especially the forests).

(3) A graph is chordal if every induced cycle is of length 3.

(1) Use the LP formulation for the nucleolus computation.

(Peleg)

By solving a sequence of LP problems, we can obtain the nucleolus (not in poly time).

(2) Identify the essential coalitions. (Huberman '80)

The essential coalitions reduce the work load.
$S \subseteq V$ is essential $\Leftrightarrow$ S is an independent set.

(3) Analyze the LP.

For O-good perfect graphs, we can precisely tell what are the optimal solutions in the LP problems with help of the characterization of the extreme points of the core.

Cor.

G chordal
$\Rightarrow$ the nucleolus of χ_G computed in poly. time.

Proof Sketch

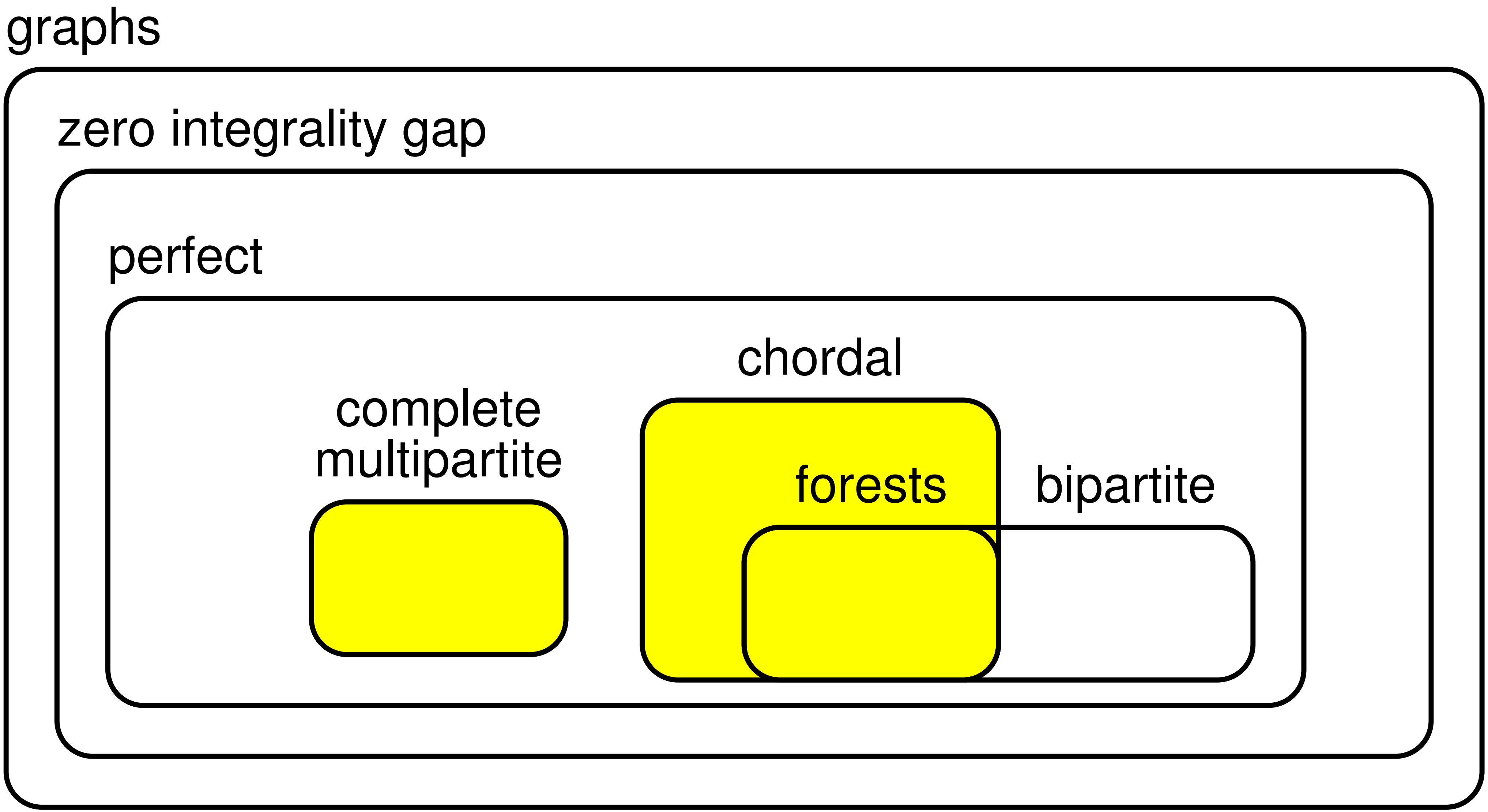
(1) Chordal graphs are O-good perfect graphs.

(2) In a chordal graph,

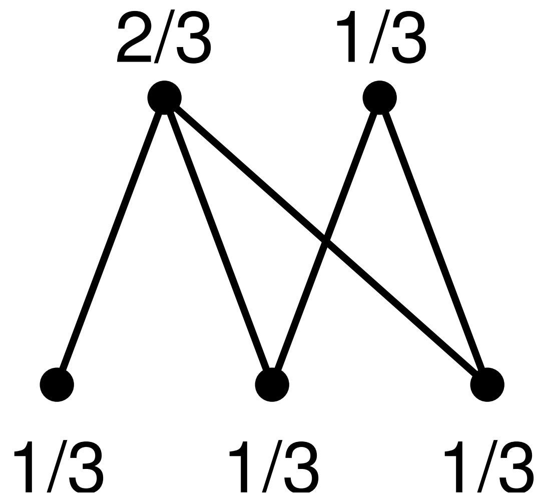
$$\# \text{ of maximal cliques} \leq \# \text{ of vertices.}$$

We can enumerate them in poly time.

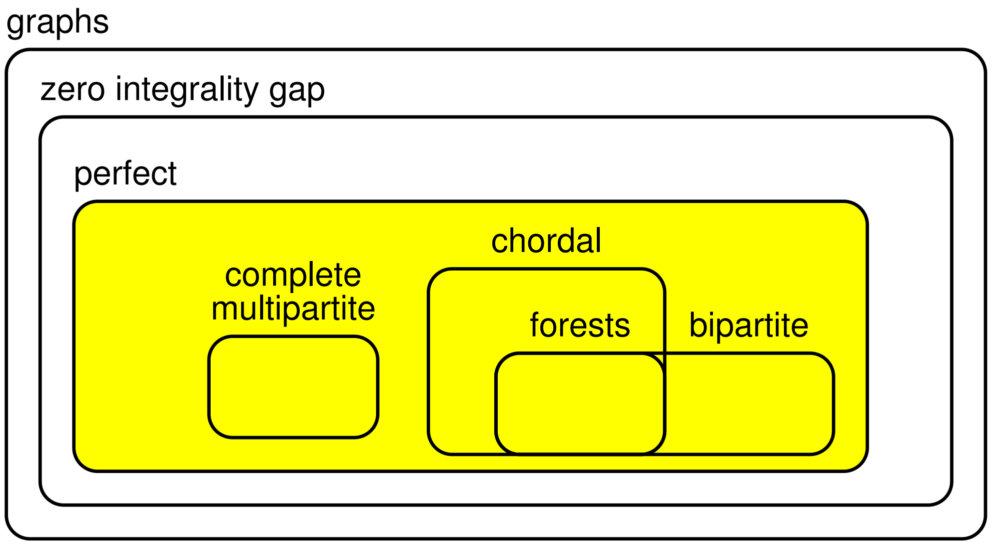
(Fulkerson & Gross '65)


graphs
zero integrality gap
perfect
chordal
complete
multipartite
forests
bipartite

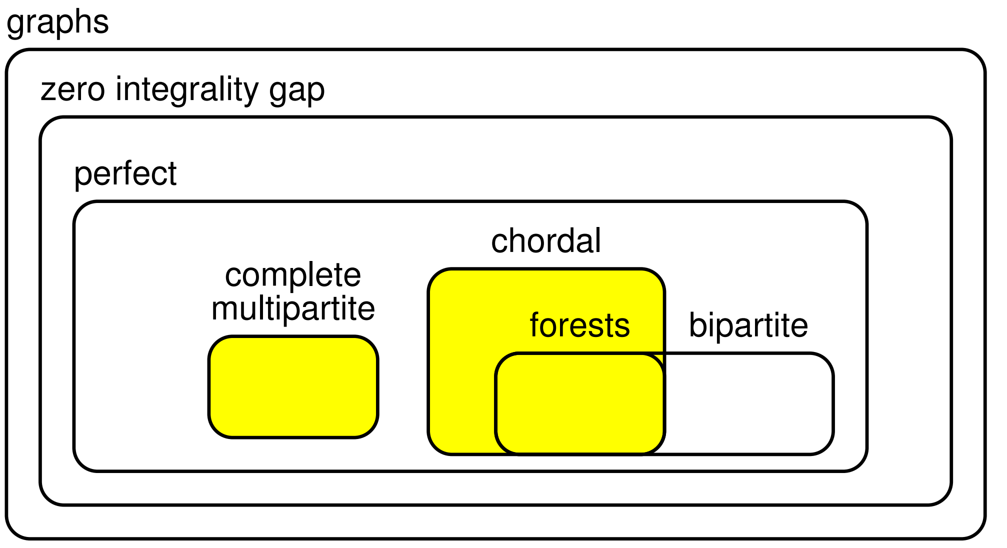
There is a bipartite graph for which the nucleolus is not the barycenter of the char. vectors of the maximum cliques.



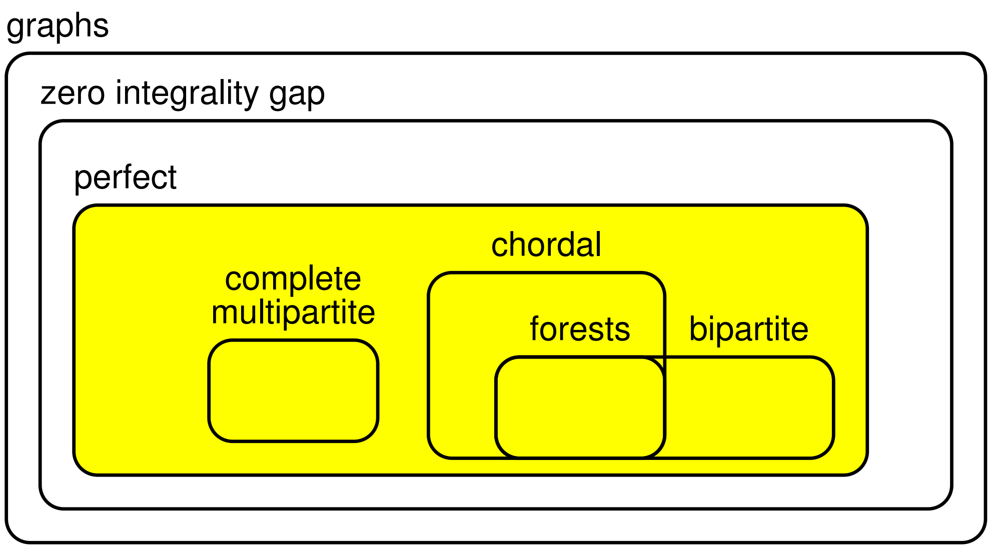
Core



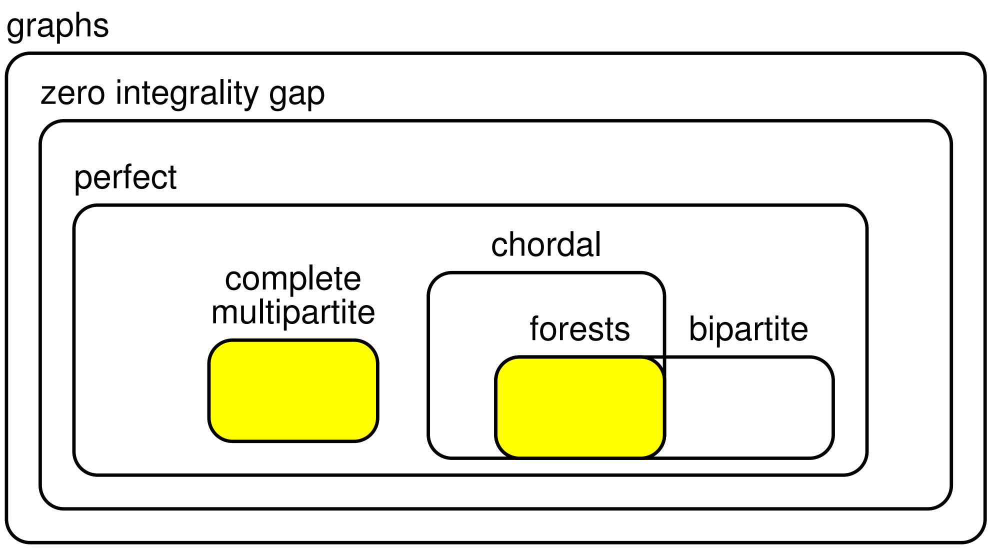
Nucleolus



τ-value



Shapley value



Computation of the nucleoli for

- perfect graphs ??
- bipartite graphs ??
- outerplanar graphs ??
- cographs ??

The list of what we discussed

- Def.: minimum coloring game
- Def.: cost allocation
- Def.: core
- Def.: nucleolus
- Characterization: the nucleolus for a chordal graph
- Open problems

Framework: Several people are willing to work together...

- They want to have a largest possible benefit.
  (optimization theory)
- They want to allocate the benefit in a fair way.
  (cooperative game theory)

Status of algorithmic problems on cooperative games

- As many cooperative games as optimization problems!!
- Many algorithmic problems remain unsolved!!

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- As many cooperative games as optimization problems!!
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$\Longrightarrow$ **Why not work on them??**

[End of the talk]

おおきに

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