Finite-Length Analysis of Irregular Expurgated LDPC Codes under Finite Number of Iterations

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The aim of our research

To estimate the bit error probability $P_b(n, \epsilon, t)$ of LDPC codes over the BEC under belief propagation decoding

where

- $n$: blocklength
- $\epsilon$: erasure probability of BEC
- $t$: the number of iterations
## Previous Results

### Analysis for the BEC

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[1] Richardson and Urbanke 2001  

### The Main Result of This Work

Our result presented in ISIT2008 is generalized for **irregular** ensembles.
Asymptotic Expansion

Asymptotic Expansion w.r.t $n$ while $t$ is fixed

$$P_b(n, \epsilon, t) = P_b(\infty, \epsilon, t) + \alpha(\epsilon, t) \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Coefficient of $1/n$

$$\alpha(\epsilon, t) := \lim_{n \to \infty} n (P_b(n, \epsilon, t) - P_b(\infty, \epsilon, t))$$

Approximation

$$P_b(n, \epsilon, t) \approx P_b(\infty, \epsilon, t) + \alpha(\epsilon, t) \frac{1}{n}$$

Our purpose is to derive $\alpha(\epsilon, t)$ for irregular ensembles.
Neighborhoods

\[ P_b(n, \epsilon, t) = \sum_{G \in \text{the set of all neighborhoods of depth } t} P_n(G) P_b(G, \epsilon) \]

\[ P_n(G) \quad \begin{array}{cccccc}
\epsilon(1-(1-\epsilon)^2)^2 & \epsilon^2(1-(1-\epsilon)^2) & \epsilon^3 & \epsilon(1-(1-\epsilon)^2) & \epsilon^2(1+\epsilon(1-\epsilon)) & \epsilon \\
\frac{(2n-6)(2n-8)}{(2n-1)(2n-5)} & \frac{2(2n-6)}{(2n-1)(2n-5)} & \frac{1}{(2n-1)(2n-5)} & \frac{2}{(2n-1)(2n-5)} & \frac{4(2n-6)}{(2n-1)(2n-5)} & \frac{2}{(2n-1)}
\end{array} \]

Order of \( P_n(G) \)

\begin{array}{ccccccc}
1 & n^{-1} & n^{-2} & n^{-2} & n^{-1} & n^{-1}
\end{array}
Number of cycles

The basic fact

If $G$ has $k$ cycles

$$\mathbb{P}_n(G) = \Theta(n^{-k}).$$

The large blocklength limit of the bit error probability

$$\mathbb{P}_b(\infty, \epsilon, t) = \lim_{n \to \infty} \mathbb{P}_n(G) \mathbb{P}_b(G, \epsilon)$$

for all $G \in$ the set of all neighborhoods of depth $t$

$$= \lim_{n \to \infty} \mathbb{P}_n(G) \mathbb{P}_b(G, \epsilon)$$

for all cycle-free neighborhoods of depth $t$. 
Calculation of $\alpha(\epsilon, t)$

$$\alpha(\epsilon, t) := \lim_{n \to \infty} n(P_b(n, \epsilon, t) - P_b(\infty, \epsilon, t))$$

$$= \lim_{n \to \infty} n \left( \sum_{G \in \text{the set of all cycle-free neighborhoods of depth } t} P_n(G)P_b(G, \epsilon) - P_b(\infty, \epsilon, t) \right)$$

$$+ \lim_{n \to \infty} n \sum_{G \in \text{the set of all single-cycle neighborhoods of depth } t} P_n(G)P_b(G, \epsilon).$$

In the previous work [Mori et al., ISIT2008], $\gamma(\epsilon, t)$ was obtained for irregular ensembles but $\beta(\epsilon, t)$ was obtained only for regular ensembles.
Contribution of Cycle-Free Neighborhoods

\[ \beta(\epsilon, t) = \]
\[
\frac{1}{2L'(1)} \left( \mathbb{E}_t[K(K - 1)P] - \sum_i \frac{i}{\lambda_i} \mathbb{E}_t[V_i(V_i - 1)P] - \sum_j \frac{j}{\rho_j} \mathbb{E}_t[C_j(C_j - 1)P] \right)
\]

The expectations are taken on the tree ensemble of depth \( t \)

\[ \mathbb{P}_\infty(G) := \lim_{n \to \infty} \mathbb{P}_n(G) \]

- \( K \): the number of edges in a tree neighborhood
- \( V_i \): the number of variable nodes of degree \( i \) in a tree neighborhood
- \( C_j \): the number of check nodes of degree \( j \) in a tree neighborhood
- \( P \): the erasure probability of the root node after BP decoding on a tree neighborhood
Method of Generating Function

\[ \mathbb{E}_t[K(K - 1)P] = \left. \frac{\partial^2 \mathbb{E}_t[x^K P]}{\partial x^2} \right|_{x = 1} \]

\[ \mathbb{E}_t[V_i(V_i - 1)P] = \left. \frac{\partial^2 \mathbb{E}_t[x^{V_i} P]}{\partial x^2} \right|_{x = 1} \]

\[ \mathbb{E}_t[C_j(C_j - 1)P] = \left. \frac{\partial^2 \mathbb{E}_t[x^{C_j} P]}{\partial x^2} \right|_{x = 1} \]

\[ \mathbb{E}_t[x^K P] = \frac{1}{x} \mathbb{E}_t \left[ \prod_k y_k^{V_k} \prod_l z_l^{C_l} P \right] \bigg|_{y_k = x, z_l = x \text{ for all } k, l} \]

\[ \mathbb{E}_t[x^{V_i} P] = \mathbb{E}_t \left[ \prod_k y_k^{V_k} \prod_l z_l^{C_l} P \right] \bigg|_{y_i = x, y_k = 1, z_l = 1 \text{ for all } k \neq i, l} \]

\[ \mathbb{E}_t[x^{C_j} P] = \mathbb{E}_t \left[ \prod_k y_k^{V_k} \prod_l z_l^{C_l} P \right] \bigg|_{z_j = x, y_k = 1, z_l = 1 \text{ for all } k, l \neq j} \]
The Mother Generating Function

\[ E_t \left[ \prod_k y_k^{V_k} \prod_l z_l^{C_l} P \right] \epsilon \mathcal{L}(F(t)), \]

where

\[ F(t) := \begin{cases} 
1, & \text{if } t = 0 \\
\mathcal{P}(g(t)) - \mathcal{P}(G(t)), & \text{otherwise},
\end{cases} \]

\[ G(t) := \mathcal{L}(f(t - 1)) - \epsilon \mathcal{L}(F(t - 1)), \]

\[ f(t) := \begin{cases} 
1, & \text{if } t = 0 \\
\mathcal{P}(g(t)), & \text{otherwise},
\end{cases} \]

\[ g(t) := \mathcal{L}(f(t - 1)), \]

and where

\[ \mathcal{L}(x) := \sum_i L_i y_i x^i, \quad \mathcal{L}(x) := \sum_i \lambda_i y_i x^{i-1}, \quad \mathcal{P}(x) := \sum_j \rho_j z_j x^{j-1}. \]
\( \alpha(\epsilon, t) \) for Optimized Irregular Ensemble

\[
\lambda(x) = 0.500x + 0.153x^2 + 0.112x^3 + 0.055x^4 + 0.180x^8
\]

\[
\rho(x) = 0.492x^2 + 0.508x^3
\]

\( R \approx 0.192, \ \epsilon_{BP} \approx 0.8, \ \ t = 1, 2, \ldots, 8, \ 50 \)
Simulation Results

\[ \lambda(x) = 0.500x + 0.153x^2 + 0.112x^3 + 0.055x^4 + 0.180x^8 \]
\[ \rho(x) = 0.492x^2 + 0.508x^3 \]

\[ R \approx 0.192, \quad \epsilon_{BP} \approx 0.8, \quad t = 20 \]
Ensembles with $\lambda_2 = 0$

$(3,6)$-regular ensemble \[ t = 5 \quad P_b(n, \epsilon, \infty) = \Theta(1/n^2) \text{ for } \epsilon < \epsilon_{BP} \]

For small $\epsilon$, the small number of iteration is sufficient unless blocklength is sufficiently large.
The Speed of Convergence

For the irregular ensemble,
\[ \lambda(x) = 0.500x + 0.153x^2 + 0.112x^3 + 0.055x^4 + 0.180x^8 \]
\[ \rho(x) = 0.492x^2 + 0.508x^3 \]
when \( t = 20, \, n = 5760, \)

\[ \alpha(\epsilon, t) \approx n \left( P_b(n, \epsilon, t) - P_b(\infty, \epsilon, t) \right) \]

for any \( \epsilon \) (Generally, \( \lambda_2 \) is larger and larger, the convergence is faster)

\( \alpha(\epsilon, t) \) consists of contributions of cycle-free neighborhoods and single-cycle neighborhoods

But the number of variable nodes in the smallest tree of depth 20 is 4194302 \( \gg \) 5760

The probability of cycle-free and single-cycle neighborhoods is zero

Open problem: Why is the speed of the convergence fast?
Conclusion and Open Problems

Conclusion

- Using the generating function method, \( \beta(\epsilon, t) \) is obtained for irregular ensembles.
- The speed of the convergence to \( \alpha(\epsilon, t) \) is fast.

Open problems

- The fast convergence to \( \alpha(\epsilon, t) \) except for ensembles with \( \lambda_2 = 0 \) and \( \epsilon \) is small.
- Minimization of \( P_b(n, \epsilon, t) + \alpha(\epsilon, t)/n \) on some conditions.
- Higher order terms i.e. coefficient of \( 1/n^2, 1/n^3 \ldots \).
- The limit parameter \( \alpha(\epsilon, \infty) \) for irregular ensembles.
- Generalization to arbitrary binary memoryless symmetric channels.
- Asymptotic analysis of performance based on other limits e.g. \( n \to \infty \) and \( t \to \infty \) simultaneously.