Conversion Methods for Large Scale SDPs to Exploit Their Structured Sparsity

The 4th Sino-Japanese Optimization Meeting

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Contents

1. Introduction
   — Semidefinite Programs (SDPs) and their conversion —

2. Two kinds of sparsities
   2-1. Aggregated sparsity and positive definite matrix completion
   2-2. Correlative sparsity and sparsity of the Schur complement matrix in SDP with small mat. variables

3. Conversion to a c-sparse LMI form SDP with small mat. variables

4. An application to sensor network localization

5. Concluding remarks
Contents

1. Introduction
   — Semidefinite Programs (SDPs) and their conversion —
2. Two kinds of sparsities
   2-1. Aggregated sparsity and positive definite matrix completion
   2-2. Correlative sparsity and sparsity of the Schur complement matrix in SDP with small mat. variables
3. Conversion to a c-sparse LMI form SDP with small mat. variables
4. An application to sensor network localization
5. Concluding remarks
Equality standard form SDP:
\[
\min \ A_0 \cdot X \ \text{sub.to} \ A_p \cdot X = b_p \ (p = 1, \ldots, m), \ S^n \ni X \succeq O
\]

Here

\[\ A_p \in S^n \ \text{the linear space of} \ n \times n \ \text{symmetric matrices} \]

with the inner product \[A_p \cdot X = \sum_{i, j} [A_p]_{ij} X_{ij}.\]

\[b_p \in \mathbb{R}, \ X \succeq O \iff X \in S^n \text{is positive semidefinite.}\]

Lots of Applications to Various Problems
- Systems and control theory — Linear Matrix Inequality
- SDP relaxations of combinatorial and nonconvex problems
  - Max cut and max clique problems
  - Quadratic assignment problems
  - Polynomial optimization problems
- Robust optimization
- Quantum chemistry
- Moment problems (applied probability)
- Sensor network localization problem — later
- . . .
Equality standard form SDP:
\[
\min A_0 \cdot X \text{ sub.to } A_p \cdot X = b_p \ (p = 1, \ldots, m), \quad S^n \ni X \succeq O
\]

SDP can be large-scale easily
- \( n \times n \) mat. variable \( X \) involves \( n(n+1)/2 \) real variables;
  \( n = 2000 \Rightarrow n(n+1)/2 \approx 2 \text{ million} \)
- \( m \) linear equality constraints or \( m \ A_p \)'s \( \in S^n \)

◊ How can we solve a larger scale SDP?

(a) Use more powerful computer system such as clusters and grids of computers — parallel computation.
(b) Develop new numerical methods for SDPs.
(c) Improve primal-dual interior-point methods.
(d) Convert a large sparse SDP to an SDP which existing pdipms can solve efficiently:
  - multiple but small size mat. variables.
  - a sparse Schur complement mat. (a coef. mat. of a sys. of equations solved at \( \forall \) iteration of the pdipm).
An SDP example — Conversion makes a critical difference

\[
\begin{align*}
\text{min} & \quad \sum_{p=1}^{m} x_p + I \cdot X \\
\text{sub.to} & \quad a_p x_p + A_p \cdot X = 2, \ x_p \geq 0 \ (p = 1, \ldots, m), \ X \succeq O.
\end{align*}
\]

Here \( a_p \in (0, 1) \) and \( A_p \in S^k \) are generated randomly.

<table>
<thead>
<tr>
<th>m</th>
<th>k</th>
<th>SeDuMi cpu time in sec.</th>
<th>conv.+SeDuMi cpu time in sec.</th>
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<tbody>
<tr>
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<td>2000</td>
<td>10</td>
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<td>10.3</td>
</tr>
<tr>
<td>4000</td>
<td>10</td>
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<td>20.9</td>
</tr>
</tbody>
</table>

\( x_p \) is an LP variable which appears in a single equality constraint.

\( X \) is an SDP variable matrix which appears in all equality constraints, and its size is small.

How can we formulate and exploit more general structured sparsity?
## Outline of the conversion

<table>
<thead>
<tr>
<th>structured sparsity used</th>
<th>a large scale and structured sparse SDP</th>
<th>technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>aggregated sparsity</td>
<td>⇓</td>
<td>positive definite mat. completion</td>
</tr>
<tr>
<td>an SDP with small SDP cones and shared variables among SDP cones</td>
<td></td>
<td></td>
</tr>
<tr>
<td>correlative sparsity</td>
<td>⇓</td>
<td>conversion to LMI form SDP or conversion to Equality form SDP</td>
</tr>
<tr>
<td>a c-sparse SDP with small mat. variables (i.e., small SDP cones)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. Introduction
   — Semidefinite Programs (SDPs) and their conversion —
2. Two Kinds of Sparsities
   2-1. Aggregated sparsity and positive definite matrix completion (Fukuda et al. ’01, Nakata et al. ’03)
   2-2. Correlative sparsity and sparsity pattern of the Schur complement matrix
3. Conversion to a c-sparse LMI form SDP with small mat. variables
4. An application to sensor network localization
5. Concluding remarks
Equality standard form SDP:
\[
\min A_0 \bullet X \text{ sub.to } A_p \bullet X = b_p \ (p = 1, \ldots, m), \ S^n \ni X \succeq O
\]

\( A_* : n \times n \) aggregated sparsity pattern mat.
\[
[A_*]_{ij} = \begin{cases} 
\star & \text{if } i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for some } p = 0, \ldots, m, \\
0 & \text{otherwise}
\end{cases}
\]

SDP : a-sparse if \( A_* \) allows a sparse Cholesky factorization

Two typical cases

1: bandwidth along diagonal

\[
A_* = \begin{pmatrix}
\star & \star & 0 & 0 & 0 \\
\star & \star & \star & 0 & 0 \\
0 & \star & \star & \star & 0 \\
0 & 0 & \star & \star & \star \\
0 & 0 & 0 & \star & \star
\end{pmatrix}
\]

2 : arrow \( \searrow \)

\[
A_* = \begin{pmatrix}
\star & 0 & 0 & 0 & \star \\
0 & \star & 0 & 0 & \star \\
0 & 0 & \star & 0 & \star \\
0 & 0 & 0 & \star & \star \\
\star & \star & \star & \star & \star
\end{pmatrix}
\]

\( X \) : fully dense, so standard \textit{pdipms} can not effectively utilize this type of sparsity \( \Rightarrow \) pos.def.mat.completion
Equality standard form SDP:
\[
\min \ A_0 \cdot X \ \text{sub.to} \ A_p \cdot X = b_p \ (p = 1, \ldots, m), \ S^n \ni X \succeq O
\]

\(A_* : n \times n \) aggregated sparsity pattern mat.

\[\left[A_*\right]_{ij} = \begin{cases} \star & \text{if } i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for some } p = 0, \ldots, m, \\ 0 & \text{otherwise} \end{cases} \]

SDP : a-sparse if \(A_*\) allows a sparse Cholesky factorization

\(G(N, E) : \) the asp graph, an undirected graph with
\(N = \{1, \ldots, n\}, \ E = \{(i, j) : [A_*]_{ij} = \star \text{ and } i < j\}. \)

\(\Downarrow \)

\(G(N, \overline{E}) : \) a chordal extension of \(G(N, E).\)

\(C_1, \ldots, C_\ell \subset N : \) the family of maximal cliques of \(G(N, \overline{E}).\)

SDP \equiv \) an SDP with shared variables among small SDP cones:

\[
\min \ \sum_{(i,j)\in\widetilde{E}} [A_0]_{ij}X_{ij} \\
\text{sub.to} \ \sum_{(i,j)\in\widetilde{E}} [A_p]_{ij}X_{ij} = b_p \ (\forall p), \ X(C_r) \succeq O \ (r = 1, \ldots, \ell),
\]

where \(X(C_r) : \) the submatrix of \(X\) consisting of \(X_{ij} \ (i, j \in C_r).\)

Here \(\widetilde{E} = \{(i, j) : (i, j), (j, i) \in \overline{E} \text{ or } i = j\} \implies \text{Section 3}.\)
Equality standard form SDP:
\[
\min A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \quad (p = 1, \ldots, m), \quad S^n \ni X \succeq O
\]

\(A_*: n \times n\) aggregated sparsity pattern mat.
\[
[A_*]_{ij} = \begin{cases} 
\star & \text{if } i = j \text{ or } [A_p]_{ij} \neq 0 \text{ for some } p = 0, \ldots, m, \\
0 & \text{otherwise}
\end{cases}
\]

SDP: \textbf{a-sparse} if \(A_*\) allows a sparse Cholesky factorization

\[
\begin{pmatrix}
\star & \star & 0 & 0 & 0 \\
\star & \star & \star & \star & 0 \\
0 & \star & \star & 0 & \star \\
0 & \star & 0 & \star & \star \\
0 & 0 & \star & \star & \star \\
\end{pmatrix}
\]

\[
G(N, E) \downarrow \quad \text{chordal max. cliques}
\]

\[
\tilde{E} = \{\star\text{'s \& 0's}\}
\]

\[
\min \sum_{(i,j) \in \tilde{E}} [A_0]_{ij} X_{ij} \quad \text{sub.to} \quad \sum_{(i,j) \in \tilde{E}} [A_p]_{ij} X_{ij} = b_p,
\]

\[
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}, \quad \begin{pmatrix}
X_{22} & X_{23} & X_{24} \\
X_{32} & X_{33} & X_{34} \\
X_{42} & X_{43} & X_{44}
\end{pmatrix}, \quad \begin{pmatrix}
X_{33} & X_{34} & X_{35} \\
X_{43} & X_{44} & X_{45} \\
X_{53} & X_{54} & X_{55}
\end{pmatrix} \succeq O
\]
Equality standard form SDP:
\[
\min \ A_0 \cdot X \ \text{sub.to} \ A_p \cdot X = b_p \ (p = 1, \ldots, m) , \ S^n \ni X \succeq O
\]

As an example: \( \downarrow \) aggregated sparsity

\[
\min \ \sum_{(i,j) \in \tilde{E}} \ [A_0]_{ij} X_{ij} \ \text{sub.to} \ \sum_{(i,j) \in \tilde{E}} \ [A_p]_{ij} X_{ij} = b_p \ \text{and} \ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} , \begin{pmatrix} X_{22} & X_{23} & X_{24} \\ X_{32} & X_{33} & X_{34} \\ X_{42} & X_{43} & X_{44} \end{pmatrix} , \begin{pmatrix} X_{33} & X_{34} & X_{35} \\ X_{43} & X_{44} & X_{45} \\ X_{53} & X_{54} & X_{55} \end{pmatrix} \succeq O
\]

(an SDP with smaller SDP cones and shared variables) \( \implies \)

Conversion into a standard form SDP to apply IPM — 2 ways

Primal form SDP with small mat. variables:

\[
\min \ "\text{linear obj. in } Y_{ij}^r \text{" sub.to } "\text{linear eq. in } Y_{ij}^r \text{" and} \ \\
\begin{pmatrix} Y_{11}^{1} & Y_{12}^{1} \\ Y_{21}^{1} & Y_{22}^{1} \end{pmatrix} , \begin{pmatrix} Y_{11}^{2} & Y_{12}^{2} & Y_{13}^{2} \\ Y_{21}^{2} & Y_{22}^{2} & Y_{23}^{2} \\ Y_{31}^{2} & Y_{32}^{2} & Y_{33}^{2} \end{pmatrix} , \begin{pmatrix} Y_{11}^{3} & Y_{12}^{3} & Y_{13}^{3} \\ Y_{21}^{3} & Y_{22}^{3} & Y_{23}^{3} \\ Y_{31}^{3} & Y_{32}^{3} & Y_{33}^{3} \end{pmatrix} \succeq O, \\
Y_{22}^{1} = Y_{11}^{2}, \ Y_{22}^{2} = Y_{11}^{3}, \ Y_{23}^{2} = Y_{12}^{3}, \ Y_{33}^{2} = Y_{22}^{3}.
\]
Equality standard form SDP:
\[ \min A_0 \bullet X \ \text{sub.to} \ A_p \bullet X = b_p \ (p = 1, \ldots, m), \ S^n \ni X \succeq O \]

As an example: \[\downarrow\] aggregated sparsity

\[ \min \sum_{(i,j) \in \tilde{E}} [A_0]_{ij} X_{ij} \ \text{sub.to} \ \sum_{(i,j) \in \tilde{E}} [A_p]_{ij} X_{ij} = b_p \ \text{and} \]

\[
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix},
\begin{pmatrix}
X_{22} & X_{23} & X_{24} \\
X_{32} & X_{33} & X_{34} \\
X_{42} & X_{43} & X_{44}
\end{pmatrix},
\begin{pmatrix}
X_{33} & X_{34} & X_{35} \\
X_{43} & X_{44} & X_{45} \\
X_{53} & X_{54} & X_{55}
\end{pmatrix}
\succeq O
\]

(an SDP with smaller SDP cones and shared variables) \[\Rightarrow\]
Conversion into a standard form SDP to apply IPM — 2 ways

LMI form SDP with small mat. variables — later
Contents

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   2-2. Correlative sparsity and sparsity of the Schur complement matrix in SDP with small mat. variables

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5. Concluding remarks
SDP with small matrix variables:

\[
\begin{align*}
\text{min} & \quad \sum_{r=1}^{\ell} A_{0r} \bullet X_r \\
\text{sub.to} & \quad \sum_{r=1}^{\ell} A_{pr} \bullet X_r = b_p \quad (p = 1, \ldots, m), \quad X_r \succeq O \quad (\forall r)
\end{align*}
\]

\[
\downarrow \quad A_{p\diamond} = \text{diag} \left( A_{p1}, \ldots, A_{p\ell} \right), \quad X_{\diamond} = \text{diag} \left( X_1, \ldots, X_\ell \right),
\]

\[
A_{p\diamond} \bullet X_{\diamond} = \sum_{r=1}^{\ell} A_{pr} \bullet X_r.
\]

SDP: \( \text{min} \ A_{0\diamond} \bullet X_{\diamond} \) sub.to \( A_{p\diamond} \bullet X_{\diamond} = b_p \quad (\forall p), \quad X_{\diamond} \succeq O \)

\( m \times m \quad R_* : \) correlative sparsity pattern (csp) mat.

\[
[R_*]_{pq} = \begin{cases} 
0 & \text{if } A_{p\diamond} \text{ and } A_{q\diamond} \text{ are bw-comp}, \\
\ast & \text{otherwise.} 
\end{cases}
\]

\( A_{p\diamond} \text{ and } A_{q\diamond} : \) block-wise complementary

\[
\uparrow \quad A_{pr} = O \text{ or } A_{qr} = O \text{ for every } r = 1, \ldots, \ell;
\]
SDP with small matrix variables:

\[
\text{min } \sum_{r=1}^{\ell} A_{0r} \bullet X_r
\]

sub.to \( \sum_{r=1}^{\ell} A_{pr} \bullet X_r = b_p \) (\( p = 1, \ldots, m \)), \( X_r \succeq O \) (\( \forall r \))

\[
A_{p\diamond} = \text{diag} \left( A_{p1}, \ldots, A_{p\ell} \right), \quad X_{\diamond} = \text{diag} \left( X_1, \ldots, X_\ell \right),
\]

\[
A_{p\diamond} \bullet X_{\diamond} = \sum_{r=1}^{\ell} A_{pr} \bullet X_r.
\]

SDP:

\[
\text{min } A_{0\diamond} \bullet X_{\diamond} \text{ sub.to } A_{p\diamond} \bullet X_{\diamond} = b_p \ (\forall p), \quad X_{\diamond} \succeq O
\]

\( m \times m \ R_* \) : correlative sparsity pattern (csp) mat.

\[
[R_*]_{pq} = \begin{cases} 0 & \text{if } A_{p\diamond} \text{ and } A_{q\diamond} \text{ are bw-comp,} \\ * & \text{otherwise.} \end{cases}
\]

\[
A_{1\diamond} = \text{diag} \left( A_{11}, O, O, O \right)
\]

\[
A_{2\diamond} = \text{diag} \left( O, A_{22}, O, O \right)
\]

\[
A_{3\diamond} = \text{diag} \left( O, O, A_{33}, O \right)
\]

\[
A_{4\diamond} = \text{diag} \left( A_{41}, A_{42}, A_{43}, A_{44} \right)
\]

\[
R_* = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{pmatrix}
\]

\( \exists \) sparse Cholesky factorization
SDP with small matrix variables:

\[
\begin{align*}
    \text{min} & \quad \sum_{r=1}^{\ell} A_{0r} \bullet X_r \\
    \text{sub.to} & \quad \sum_{r=1}^{\ell} A_{pr} \bullet X_r = b_p \ (p = 1, \ldots, m), \quad X_r \succeq O \ (\forall r)
\end{align*}
\]

\[
A_{p\diamond} = \text{diag} \left( A_{p1}, \ldots, A_{p\ell} \right), \quad X_\diamond = \text{diag} \left( X_1, \ldots, X_\ell \right), \quad A_{p\diamond} \bullet X_\diamond = \sum_{r=1}^{\ell} A_{pr} \bullet X_r.
\]

SDP: \(\text{min } A_{0\diamond} \bullet X_\diamond \text{ sub.to } A_{p\diamond} \bullet X_\diamond = b_p \ (\forall p), \quad X_\diamond \succeq O\)

\(m \times m \ R_* : \text{correlative sparsity pattern (csp) mat.}\)

\[
[R_*]_{pq} = \begin{cases} 
0 & \text{if } A_{p\diamond} \text{ and } A_{q\diamond} \text{ are bw-comp,} \\
* & \text{otherwise.}
\end{cases}
\]

\(\bullet R_* = \text{the sparsity pattern of the Schur complement mat.} = \)
\(\text{a coef. mat. of equations solved at } \forall \text{ iteration of the pdipm by the Cholesky fact.}\)

SDP : c-sparse if \(R_*\) allows a sparse Cholesky factorization

c-sparse SDP with small mat. variables — target of conversion
## Outline of the conversion

<table>
<thead>
<tr>
<th>Structured Sparsity Used</th>
<th>A Large Scale and Structured Sparse SDP</th>
<th>Technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregated Sparsity</td>
<td>↓</td>
<td>Positive Definite Mat. Completion</td>
</tr>
<tr>
<td></td>
<td>An SDP with small SDP cones and shared variables among SDP cones</td>
<td></td>
</tr>
<tr>
<td>Correlative Sparsity</td>
<td>↓</td>
<td>Conversion to LMI Form SDP or Conversion to Equality Form SDP</td>
</tr>
<tr>
<td></td>
<td>↓</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A c-sparse SDP with small mat. variables (i.e., small SDP cones)</td>
<td></td>
</tr>
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</table>

\[ \quad \]
Contents

1. Introduction
   — Semidefinite Programs (SDPs) and their conversion —
2. Two kinds of sparsities
   2-1. Aggregated sparsity and positive definite matrix completion
   2-2. Correlative sparsity and sparsity of the Schur complement matrix in SDP with small mat. variables
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SDP with shared variables among SDP cones

\[
\begin{align*}
\min & \sum_{(i,j) \in \widetilde{E}} [A_0]_{ij} X_{ij} \quad \text{sub.to} \quad \sum_{(i,j) \in \widetilde{E}} [A_p]_{ij} X_{ij} = b_p \ (p = 1, \ldots, m), \\
& \sum_{(i,j) \in \widetilde{E}} X(C_r) \succeq O \ (r = 1, \ldots, \ell), \\
C_1, \ldots, C_r : \text{the max. cliques of a chordal graph } G(N, \overline{E})
\end{align*}
\]

\[\widetilde{E} = \{(i, j) : (i, j), (j, i) \in \overline{E} \text{ or } i = j\}.\]

Represent each \(X(C_r)\) as

\[
X(C_r) = \sum_{i,j \in C_r, i \leq j} E_{ij}(C_r) X_{ij},
\]

where \(E_{ij}(C_r)\) : a sym. mat. with 1 at some one or two elements and 0 elsewhere. Then, a \(c\)-sparse LMI form SDP having eq. const.

\[
\begin{align*}
\min & \sum_{(i,j) \in \widetilde{E}} [A_0]_{ij} X_{ij} \quad \text{sub.to} \quad \sum_{(i,j) \in \widetilde{E}} [A_p]_{ij} X_{ij} = b_p \ (\forall p), \\
& \sum_{i,j \in C_r, i \leq j} E_{ij}(C_r) X_{ij} \succeq O \ (\forall r).
\end{align*}
\]
SDP with shared variables among SDP cones

\[
\begin{align*}
\min & \sum_{(i,j) \in \tilde{E}} [A_0]_{ij} X_{ij} \quad \text{sub.to} \quad \sum_{(i,j) \in \tilde{E}} [A_p]_{ij} X_{ij} = b_p \quad (p = 1, \ldots, m), \\
& X(C_r) \succeq \mathbf{0} \quad (r = 1, \ldots, \ell), \\
C_1, \ldots, C_r : & \text{the max. cliques of a chordal graph } G(N, \overline{E}) \\
\tilde{E} & = \{(i, j) : (i, j), (j, i) \in \overline{E} \text{ or } i = j\}.
\end{align*}
\]

\(n = 100, m = 98, C_r = \{r, 99, 100\} \quad (1 \leq r \leq 98)\).

\[A_r = \begin{bmatrix}
      & \ast & \ast &  \ast  \\
\ast & \ast & \ast  & \ast  \\
\ast & \ast & \ast & \ast  \\
\ast & \ast & \ast & \ast  \\
\end{bmatrix}\]

\[A_* = \begin{bmatrix}
      & \ast & \ast &  \ast  \\
\ast & \ast & \ast  & \ast  \\
\ast & \ast & \ast & \ast  \\
\ast & \ast & \ast & \ast  \\
\end{bmatrix}\]

\[R_* \text{ of LMI form SDP} = \]

\[\begin{bmatrix}
      & \ast & \ast &  \ast  \\
\ast & \ast & \ast  & \ast  \\
\ast & \ast & \ast & \ast  \\
\ast & \ast & \ast & \ast  \\
\end{bmatrix}\]
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Sensor network localization problem: Let $s = 2$ or $3$.

- $\mathbf{x}^p \in \mathbb{R}^s$ : unknown location of sensors ($p = 1, 2, \ldots, m$),
- $\mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^s$ : known location of anchors ($r = m + 1, \ldots, n$),
- $d_{pq} = \|\mathbf{x}^p - \mathbf{x}^q\| + \epsilon_{pq}$ — given for $(p, q) \in \mathcal{N}$,
- $\mathcal{N} = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a given radio range}\}$

Here $\epsilon_{pq}$ denotes a noise.

$m = 5$, $n = 9$.
1, $\ldots$, 5: sensors
6, 7, 8, 9: anchors

Anchors’ positions are fixed.
A distance is given for $\forall$ edge.
Compute locations of sensors.

$\Rightarrow$ Some nonconvex QOPs
- SDP relaxation +? — FSDP by Biswas-Ye ’06, ESDP by Wang et al ’07, ... for $s = 2$.
- SOCP relaxation — Tseng ’07 for $s = 2$.
- ...

...
Numerical results on 4 methods (a), (b), (c) and (d) applied to a sensor network localization problem with

\[ m = \text{the number of sensors dist. randomly in } [0, 1]^2, \]

4 anchors located at the corner of \([0, 1]^2\),
\[ \rho = \text{radio distance} = 0.1, \text{ no noise}. \]

(a) FSDP  
(b) FSDP + Conv. to LMI form SDP, as strong as (a)  
(c) FSDP + Conv. to equality form SDP as strong as (a)  
(d) ESDP — a further relaxation of FSDP, weaker than (a);

<table>
<thead>
<tr>
<th>( m )</th>
<th>SeDuMi cpu time in second</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a)</td>
</tr>
<tr>
<td>500</td>
<td>389.1</td>
</tr>
<tr>
<td>1000</td>
<td>3345.2</td>
</tr>
<tr>
<td>2000</td>
<td>111.1</td>
</tr>
<tr>
<td>4000</td>
<td>182.1</td>
</tr>
</tbody>
</table>

SeDuMi parameters
\[ \text{pars.free}=0; \]
\[ .\text{eps}=1.0\text{e}-5 \]
\[ \Rightarrow \text{a-sparsity, c-sparsity} \]
\[ \text{in (a) and (b)} \]
A sensor network localization problem with 1000 sensors dist. randomly in $[0, 1]^2$, 4 anchors located at the corner of $[0, 1]^2$, $\rho = \text{radio distance} = 0.1$, no noise

(b) FSDP + Conversion to an LMI form SDP
A Cholesky fact. of the a-sparsity pattern matrix $A_*$ with the symm. min. deg. ordering

(a) FSDP (Biswas-Ye ’06)   (b) FSDP + Conversion to an LMI form SDP

$1002 \times 1002$, $nz = 7062$
$\text{nz density} = 0.014$

$7381 \times 7381$, $nz = 37,701$
$\text{nz density} = 0.0014$
A Cholesky fact. of the c-sparsity pattern matrix $R_*$ (= the Schur comp. matrix) with the symm. min. deg. ordering

(a) FSDP (Biswas-Ye ’06)  \hspace{1cm} (b) FSDP + Conversion to an LMI form SDP

3686 × 3686, nz = 6,795,141  \hspace{1cm} 8916 × 8916, nz = 805,183

nz density = 1.00  \hspace{1cm} nz density = 0.020

3345.2 second  \hspace{1cm} 60.4 second
1. Introduction  
   — Semidefinite Programs (SDPs) and their conversion —  
2. Two kinds of sparsities  
   2-1. Aggregated sparsity and positive definite matrix completion  
   2-2. Correlative sparsity and sparsity of the Schur complement matrix in SDP with small mat. variables  
3. Conversion to a c-sparse LMI form SDP with small mat. variables  
4. An application to sensor network localization  
5. Concluding remarks
1. Conversion of a large scale SDP into an SDP having small mat. variables and a sparse Schur complement mat. by exploiting the structured sparsity,
   - aggregated sparsity,
   - correlative sparsity.

2. Two different methods:
   - Conversion to an LMI form SDP.
   - Conversion to an equality form SDP

3. An application to sensor network localization.
   ⇒ S. Kim’s talk on Aug. 30.

Thank you!